

# Splitting Fields

Prove the following. Let  $K$  be a field.

1. Let  $f \in K[x]$  be a polynomial of degree  $n \geq 1$ . Then there is an extension  $L : K$  of degree at most  $n$  such that  $f$  has a root in  $L$ .
2. Let  $f \in K[x]$  be a polynomial of degree  $n \geq 1$ . Then there is an extension  $M : K$  of degree at most  $n!$  such that  $f$  *splits completely over*  $M$  i.e. it factors as a product of linear polynomials of  $M$ . Equivalently,  $f$  has exactly  $n$  roots in  $M$ , counted with multiplicities.

**Definition:** Let  $f \in K[x]$ . A finite extension  $L : K$  is called a **splitting field** of  $f$  over  $K$  if  $f$  splits completely over  $L$  but not over any proper subfield.

Note: A splitting field exists, by Problem (2) above. We'll prove later that it's unique up to isomorphism.

3. Let  $f(x)$  be the minimum polynomial of  $\sqrt{3} + \sqrt{5}$  over  $\mathbb{Q}$ . Describe a splitting field of  $f(x)$  over  $\mathbb{Q}$ . What is the degree of the extension?  
(Hint: prove that  $\pm\sqrt{3} \pm \sqrt{5}$  are roots of  $f(x)$ .)
4. Let  $K$  be the smallest subfield of  $\mathbb{C}$  containing all the 8 roots of  $x^8 - 2$ , i.e. a splitting field of  $x^8 - 2$  over  $\mathbb{Q}$ . We will compute  $[K : \mathbb{Q}]$ . (Any guesses?)
  - (a) Prove that  $K = \mathbb{Q}(\sqrt[8]{2}, \omega)$ , where  $\omega = e^{\frac{2\pi i}{8}} \in \mathbb{C}$ .
  - (b) Find the minimum polynomial of  $\omega$  over  $\mathbb{Q}$ .
  - (c) Prove that  $\mathbb{Q}(\omega) = \mathbb{Q}(\sqrt{2}, i)$ .
  - (d) Prove that  $K = \mathbb{Q}(\sqrt[8]{2}, i)$ .
  - (e) What is  $[K : \mathbb{Q}]$ ?

(Compare with Example 5.4 in Howie's book, on  $x^4 - 2$ .)