The Fundamental Theorem of Galois Theory (part 2)

Definitions:
For $L \supseteq E \supseteq K$, define $\Gamma(E) = \mathrm{Gal}(L : E) = \{\sigma \in \mathrm{Gal}(L : K) \mid \sigma(y) = y \text{ for all } y \in E\}$.
For $H \leq \mathrm{Gal}(L : K)$, define $\Phi(H) = \{y \in L \mid \sigma(y) = y \text{ for all } \sigma \in H\}$.

The Fundamental Theorem of Galois Theory
Let L be a finite Galois extension of K.

(i) The maps Φ and Γ are inverses of each other.

(ii) Suppose $L \supseteq E \supseteq K$. The extension $E : K$ is Galois if and only if $\mathrm{Gal}(L : E)$ is a normal subgroup of $\mathrm{Gal}(L : K)$. In this case,

$$\mathrm{Gal}(E : K) \cong \mathrm{Gal}(L : K) / \mathrm{Gal}(L : E).$$

Proof of $\Rightarrow$ direction in (ii). Suppose $E : K$ is Galois. We will find a group homomorphism from ____________to ____________whose kernel is ____________.

Let $\sigma \in \mathrm{Gal}(L : K)$. **Claim:** $\sigma(E) = E$.

Proof of Claim: Since E is Galois over K, there exists an element $\alpha \in E$ such that $E = K(\alpha)$, whose minimal polynomial $p(x) \in K[x]$ is separable and splits over E. Consider the action of σ on the roots of $p(x)$ and complete the proof:

$\square$

Then $\sigma|_E \in \mathrm{Gal}(E : K)$ because

Let $\varphi :$ ____________ $\rightarrow$ ____________ be a map defined by $\sigma \mapsto$ ______.

Check that φ is a group homomorphism with the desired kernel:

And φ is surjective because

Proof of $\Leftarrow$ direction in (ii). Now suppose that $\mathrm{Gal}(L:E)$ is a normal subgroup of $\mathrm{Gal}(L:K)$. We will show that $E:K$ is Galois by showing that E is the splitting field of a separable polynomial over K.

Claim: For every $\sigma \in \mathrm{Gal}(L:K)$, we have $\sigma(E) \subseteq E$.

Proof of Claim: Let $\sigma \in \mathrm{Gal}(L:K)$. Since ____________is a normal subgroup of ____________,

$$\sigma^{-1}\underline{\hspace{3cm}}\sigma = \underline{\hspace{3cm}}.$$

Then for any $a \in E$ and for any $\tau \in \mathrm{Gal}(L:E)$, $\sigma^{-1}\tau\sigma(a) =$ ______ because

so $\tau\sigma(a) =$ ____________, which implies that $\sigma(a) \in$ __________________ because

$\square$

There exists a finite set $A \subset E$ such that $E = K(A)$, because

Let $B = \{\sigma(a) \mid a \in A \text{ and } \sigma \in \mathrm{Gal}(L:K)\}$. Let $f(x) = \prod_{b \in B}(x-b)$. Then $f(x) \in$ ____________ because

But $B \subseteq E$ because

Complete the proof: