Finitely Generated Modules over a PID

We will only deal with commutative unital rings today.
Lemma. Let $\varphi : M \to N$ be an R-module homomorphism. If the kernel and image of φ are finitely generated, then M is also finitely generated.
Proof.

Definition. A ring R is called *Noetherian* if every ideal in R is finitely generated.
Example: A polynomial ring in an infinite number of variables $k[x_1, x_2, x_3, \ldots]$ is not Noetherian.

Example: A principal ideal domain is Noetherian.

Theorem. Over a Noetherian ring R, any submodule of free module R^n is finitely generated.
Proof. We will proceed by induction on n.
$n = 1$ case:

Suppose $n \geq 2$. Let $\pi : R^n \to R^{n-1}$ be the projection map $(r_1, \ldots, r_n) \mapsto (r_1, \ldots, r_{n-1})$. Then $\ker(\pi) \cong$ ______ as an R-module. (Complete the proof using the previous Lemma).

Corollary. Over a Noetherian ring R, any submodule of a finitely generated module is finitely generated.
Proof. Let M be a finitely generated module over R. Then there is a surjective homomorphism $\varphi : R^n \to M$ for some integer n. Let $N \subset M$ be a submodule, and consider $\varphi^{-1}(N)$.

Let M be a finitely generated module over a Noetherian ring R. Show that there exists an $m \times n$ matrix A with entries in R (called a *presentation matrix*) such that $M \cong R^n / \operatorname{row}(A)$, where $\operatorname{row}(A)$ denotes the R-submodule of R^n spanned by rows of A.

Which rows and column operations can you do without changing the structure of the module?

Example Transform the following matrix to a "diagonal" matrix without changing the structure of the corresponding $\mathbb{Z}$-module: $\begin{pmatrix} 2 & -2 & 4 & 2 \\ -4 & 7 & 1 & 11 \\ 2 & -5 & 7 & -25 \end{pmatrix}$