GUIDELINES FOR GOOD MATHEMATICAL WRITING

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Communicating mathematics well is an important part of doing mathematics. Whether you are speaking or writing, learning to communicate effectively is not just a service to your audience; it is also an exercise in clarifying and structuring your own thinking. Moreover, beyond developing a basic competence, there is an art and elegance to good writing that every writer should strive for. And writing, as a work of art, can bring great personal satisfaction.

These guidelines may serve as a starting point for good mathematical writing.

1. BASICS

Know your audience. This is the most important consideration for writers. Put yourself in your reader's shoes. What background can we assume of the reader? What terminology should we define? What kind of "voice" do we want to project: casual or professional, serious or inviting, terse or loquacious?

If you are a student writing solutions for a homework set and your professor has not specified your audience, a good rule of thumb is to assume you are writing to another student in the course who has not yet done the assignment. Though you may assume that she has attended all the same lectures and has read the same textbook, it is standard courtesy to remind your reader of any relevant items that she has recently learned in class or from the textbook, or things she should know but might have forgotten.

For instance, if the concept of a rational number was only recently learned in class, you might insert "Recall that a rational number can be expressed as a fraction;" before saying: "since x is rational, we can set $x = m/n$ where m and n are integers."

Set an invitational tone. It is traditional to create an inviting atmosphere in one's mathematical writing. In effect, we invite readers to join us in our reasoning process by writing in the present tense, using the pronoun "we" instead of "I" (e.g., "we construct a tangent plane..."), and directing the reader with gentle commands (e.g., "let n be...", "recall that...", or "consider the set of...").

Use complete sentences. All mathematics should be written in complete sentences. Open any mathematics text and you'll see that this is true. Equations, even displayed ones, have punctuation that help you see where it fits in the context of a larger sentence. Consider this piece of writing:

$$(x-2)^2 + (x-1)^2 = 5^2 \qquad 5^2 = 25$$
$$(x-2)^2 = x^2 - 4x + 4 + x^2 - 2x + 1 = 25$$
$$2x^2 - 6x - 20$$
$$2(x+2)(x-5) \quad x = -2, 5 \quad x > 0 \quad x = 5$$

Can you figure out what the writer is doing? What's being assumed? What's being proved? Where does one thought end and another begin? What's the relationship between these phrases? Some phrases are dangling, and others, as statements, are not even true. The reader should not have to figure out what the writer was thinking!

Now consider the work of another writer who has attempted the same problem:

Date: January 19, 2011.

> **Problem.** Find a point on the line $y = x$ which is distance 5 from the point $(2, 1)$ and such that $x > 0$.
>
> **Solution.** We wish to solve $(x - 2)^2 + (x - 1)^2 = 5^2$, an equation obtained from the distance formula in the plane. Some algebra turns this equation into:
> $$2x^2 - 6x - 20 = 0.$$
> Factoring the left side, we obtain
> $$2(x + 2)(x - 5) = 0,$$
> whose solutions are evidently $x = -2$ and $x = 5$. Since we assumed $x > 0$, we have $(5, 5)$ as the desired point on the line $y = x$.

Here, the writer has clearly stated the problem and described her path to a solution. She has set an invitational tone, and every thought is expressed in a complete sentence. Now it is clear that $x > 0$ is a condition, not a result. Notice the punctuation in equations: one ended with a period because her thought was complete, the other ended with a comma because she wanted to continue the thought. Since she assumed her audience could do algebra, she didn't bore them with trivial algebraic manipulation, which would only obscure the thread of her arguments. But she did show the most interesting parts: the resulting polynomial and its factoring. And she made sure she answered the original question.

Avoid shorthand in formal writing. The many types of mathematical writing can be loosely grouped into formal and informal writing. Informal writing includes writing on a blackboard during lecture, or explaining something to a friend on a piece of scratch paper. Formal writing includes the kind of writing expected on a homework assignment, or in a paper. There are differences in what is acceptable.

For instance, in informal writing, it is common to use shorthand for quantifiers and logical connectives: symbols such as $\forall, \exists, \implies, \iff$, or abbreviations like "iff" and "s.t.".

However, in formal writing such shorthand should generally be avoided. You should write out "for all", "there exists", "implies", "if and only if", and "such that". Most other symbols are acceptable in formal writing, after defining them where needed. The membership symbol $\in$ ("is an element of") is traditionally acceptable in formal writing, as are relations (e.g., $<, >, \times, +, \cup, \cap$, etc.), variable names (e.g., x, y, z, etc.), and symbols for sets ($\mathbb{Q}, \mathbb{R}, \mathbb{Z}$, etc.). Here is an acceptable use of symbols in formal mathematical writing.

> Let A, B be two subsets of $\mathbb{R}$. We say A *dominates* B if for every $x \in A$ there exists $y \in B$ such that $y > x$.

Learn the etiquette. The above example also illustrates two common conventions of mathematical etiquette. It is customary to avoid beginning sentences or phrases with a number or symbol; 41 people told me not to do it because it looks bad. It is also customary to *emphasize* unfamiliar words that we are about to define, either by italicizing or underlining them. Other rules of etiquette can be learned through observing the norms of the mathematical people or textbooks in your area of study.

2. Towards Elegance

Decide what's important to say. Writing well does not necessarily mean writing more. If your solution is too wordy, it can sometimes obscure the points you are making.

A well-written solution will present just enough details and highlight the most interesting or unexpected parts of the argument. What theorems or axioms were crucial in getting your solution, and where were they used? Your role as a writer is not primarily to communicate details (though that can be quite important); your primary role is to give *insight*.

Highlight structure. If your argument is going to be a long one, with lots of technical details, then try to help the reader by summarize the outline of your argument at the beginning. Then, throughout your writing, help your reader see how you are progressing through your outline.

Use paragraphs to emphasize blocks of ideas that are related. The role of the first sentences of paragraphs are crucial: imagine a reader skimming your writing and only reading the first sentences. Will they see the flow of your argument? Similarly, you might want to display only the most important equations. Replacing an oft-repeated argument by a good lemma can help streamline the flow of your argument, as well as highlight a key idea.

Choose good examples. A difficult idea may be easier to digest if accompanied by an example. Choose one simple enough to follow, but interesting enough to retain the salient features. A proof of a very general idea could be preceded by an example in a specific context. A long exposition might benefit from a running example— one in which the same example is used multiple times in different contexts.

Avoid red herrings! Omit details that have no bearing on the solution of the problem, as they may throw the reader off. For example, if you say "we express the rational r as m/n where m and n have no common factors," you are leaving a clue that later, you will use the "no common factors" idea. So if you never use that fact, you should omit saying it. It's extraneous.

Step back and simplify! After writing a proof, step back and ask: how can I simplify this argument? Did I use every tool I pulled out to solve this problem? Can I streamline this argument? For example, consider this apparent proof by contradiction:

Problem. Show that if 4 divides an integer n, then n is even.

Proof (by contradiction). Suppose n were not even.
Since 4 divides n, we have $n = 4k$ for some integer k. Thus $n = 2(2k)$, which is even. This contradicts our hypothesis that n is not even. □

Do you see why this is not really a proof by contradiction? The contradictory hypothesis in the first sentence was never used! Strip away the first and last sentence, and you have an elegant, direct proof.

Refine, refine, refine. Good writing is a process of successive approximations. You should not expect your first draft to be perfect. You will find that when you review your writing, you will see ways to shorten an argument, or say something in a better way. This is the part of the writing process that will help clarify your own thinking as well.

Often, after completing a draft, a writer may notice that a particular choice of notation or a choice of definition was not optimal. A lazy writer would leave things as they are, but a thoughtful writer will take the time to go back and make changes.

3. FINALLY...

Observe the culture. Good communication is inseparable from the culture in which it takes place. This realization may unsettle budding mathematicians who are attracted to the logical absolutes of mathematics. But even these absolutes are expressed in different ways by mathematicians of different eras, as can be seen by comparing Newton's writings to any of today's calculus texts. The rules of mathematical etiquette have evolved over time.

While these guidelines attempt to draw up some common principles for formal writing, there will always be exceptions— because some subset of mathematicians may have a slightly different norm. The best rule of thumb to get a sense of what is acceptable or conventional in your cultural context is to browse several highly regarded textbooks or papers related to the document you are writing.