Name: ________________________

Math 2106 – Final Exam
December 12, 2016

Justify all your answers. Unless otherwise noted, you may use the results proven in class, in the book, or in the homework. Clearly write down the statements you use.

Do 6 out of 7 problems. Circle the problems to be graded: 1 2 3 4 5 6 7

1. Let A, B, and C be sets. Prove that $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subset A$.

2. Let n be an integer ≥ 2.

   (a) What is $\mathbb{Z}_n$? What is $\mathbb{Z}_n^*$?

   (b) Prove that the multiplication in $\mathbb{Z}_n$ defined by $[a][b] = [ab]$ is well-defined.

   (c) Prove that if n is not prime, then $\mathbb{Z}_n^*$ is not a group under multiplication.

3. Prove or disprove each of the following.

   (a) The function $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = 3x + 2$ is bijective.

   (b) The function $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(x) = 3x + 2$ is bijective.

   (c) Let A, B, and C be sets and $f : A \to B$ and $g : B \to C$ be functions. If $g \circ f$ is onto, then both f and g are onto.

4. (a) Prove that $3 \mid a$ if and only if $3 \mid a^2$ for any integer a.

   (b) Prove that $\sqrt{3}$ is irrational.

5. Prove that $|2^{\mathbb{N}\times\mathbb{N}}| = |2^{\mathbb{N}} \times 2^{\mathbb{N}}|$.

6. Let H and K be two subgroups of a group G.

   (a) Prove that $H \cap K$ is a subgroup of G.

   (b) Give a counterexample to show that $H \cup K$ is not always a subgroup of G.

   (Extra credit: Show that if $H \cup K$ is a subgroup of G, then $H \subseteq K$ or $K \subseteq H$.)

7. (a) Prove that $\sum_{i=1}^{n} r^i = \frac{r-r^{n+1}}{1-r}$ for any real number $r \neq 1$ and natural number $n \geq 1$.

(b) Prove using the definition of convergence that the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$ converges.