Sets

We do not define sets, but instead we define what operations can be done on sets. We can talk about whether an object x belongs to a set A (notation: $x \in A$.). All the variables on this worksheet refer to sets. We use the word "collection" and "set" interchangably.

Some Axioms of (Zermelo–Fraenkel) Set theory

Extension Two sets are equal if and only if they have the same elements.
(Note: Being "equal" is a logical concept, rather than a set theory concept. Two objects and x are y equal if $P(x)$ and $P(y)$ have the same truth values for all statements P.)

Specification If A is a set and $P(x)$ is a statement, then $\{x \in A : P(x)\}$ is also a set.

Unordered Pairs For any two sets there exists a set that they both belong to.

Unions For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.

(There are a few more axioms. We will look at them in the next couple of classes.)

Now take a deep breath and forget everything you know about mathematics (for the rest of the class period). Forget about circles and squares and sines and cosines. Forget about natural numbers and real numbers. Forget about addition and multiplication. Forget about functions. We will try to rebuild the mathematical universe using only logic and some axioms about sets.

1. Assume that a set exists. Use the Axiom of Specification to show that there exists a set containing no elements.
2. Use the Axiom of Extension to show that the set consisting of no elements is unique. We will denote it by $\varnothing$.
3. Use the Axiom of Unordered Pairs to show that $\{\varnothing\}$ is a set. Is it equal to $\varnothing$?
4. Use the Axioms above to prove that: If A and B are sets, then $\{A, B\}$ is a set.
5. Use the Axioms above to prove that "For every collection of sets there exists a set that contains **exactly** all the elements that belong to at least one set of the given collection." (That is, the union of sets is a set.)
6. Come up with sets containing one, two, three, four, and five elements each. Can you find a systematic way to keep going?
7. Do we really need the Axiom of Unions? If a and b are sets, then $\{x : x \in a \text{ or } x \in b\}$ is also a set by the Axiom of Specification, right? (More generally, if A is a collection of sets, then isn't $\{x : x \in a \text{ for some } a \in A\}$ also a set?)
8. Why don't we need an "Axiom of Intersections"? Prove that for any non-empty collection of sets, there exists a set consisting exactly of all the elements that belong to all sets of the given collection. (That is, the intersection of sets is a set.)
9. Can we show that the power sets exist from the above axioms?