Set Theory II

Last time we discussed the Axioms of Extension, Specification, Unordered Pairs, and Unions.

Some more aioms of set theory

Powers For each set there exists a collection of sets that contains among its elements all the subsets of the given set. (Combined with the Axiom of Specification, it follows that if A is a set, then $\mathcal{P}(A) = \{B : B \subseteq A\}$ is also a set.)

Regularity Every non-empty set contains an element that is disjoint from itself.

Notes:

- The Axiom of Regularity is not in Halmos' book, and much of set theory can be developed without it, but I am including it here to discuss the issue of a set being an element of itself.
- We are omitting the *Axiom of Replacement*, which says that the image of a set under a function is a set. We will discuss functions in much more details later.

1. (Russell paradox) Prove that "There is no Universe" without using the Axiom of Regularity. (Let A be a set and $B = \{x \in A : x \notin x\}$. Prove that $B \notin A$.)
2. Write the Axiom of Regularity in symbolic logic.
3. Prove that if we assume the Axiom of Regularity (and the four axiom from last class), then a set cannot be an element of itself.
(Let A be a set. Apply the Axiom of Regularity to $\{A\}$.)
4. Prove that "There is no Universe" using the Axiom of Regularity.
5. Explain why the intersection of an empty collection of sets is undefined.
6. Let A and B be sets. Explain how the Cartesian product $A \times B$ can be defined so that its existence follows from the Axioms we have seen.