Natural Numbers

We define $0 = \varnothing$. For any set x, its **successor** is defined as $x^+ = x \cup \{x\}$.

A set A is called a **successor set** (or an inductive set) if $0 \in A$ and $x^+ \in A$ for every $x \in A$.

Axiom of Infinity: There exists a successor set.

Exercises

1. Describe a successor set. Write down a few elements in it. Describe a different successor set.
2. Let A be a successor set. Prove that the intersection of all successor subsets of A is a successor set itself.
3. Prove that there exists a unique successor set that is a subset of every successor set. (Note: If B is any successor set, then $B \cap A$ is a successor set.)

Definition of natural numbers
The unique minimal successor set (from Question 3 above) is called the *set of natural numbers* and is denoted $\mathbb{N}$ or ω. A *natural number* is defined to be an element of $\mathbb{N}$.

Properties of $\mathbb{N}$ (The Peano Axioms)

(I) $0 \in \mathbb{N}$

(II) $\forall n \in \mathbb{N}, n^+ \in \mathbb{N}$

(III) $\forall S \subset \mathbb{N}, [0 \in S \wedge (\forall n \in S, n^+ \in S)] \Rightarrow S = \mathbb{N}$.

(IV) $\forall n \in \mathbb{N}, n^+ \neq 0$.

(V) $\forall n, m \in \mathbb{N}, (n^+ = m^+ \Rightarrow n = m)$.

Properties (I)-(IV) follow easily from the definition of $\mathbb{N}$, and (V) can also be proven from the definition. (See Halmos.) Property (III) is called the *Principle of Mathematical Induction.*

Addition Define $x + 1 = x^+$, $x + 2 = (x + 1)^+$, $x + 3 = (x + 2)^+, \ldots, x + (n + 1) = (x + n)^+$, and so on inductively. (See the "Recursion Theorem" in Section 12 of Halmos for details.)

Multiplication Define $2x = x + x$, $3x = 2x + x, \ldots, (n + 1)x = nx + x$, and so on.

Ordering For natural number m and n, we say $m < n$ if ____________.

Exercises

4. For natural numbers m and n, what does $m \cap n$ and $m \cup n$ mean in terms of the usual operations on natural numbers?
5. The **Well-ordering Principle** (WOP) says that every non-empty subset of natural numbers has a smallest element.
   (a) Write WOP in symbolic logic.
   (b) Prove WOP using the Peano Axioms.