Exam 2 review
Sets, Induction, Relations

Know how to:

- prove or disprove $A \subseteq B$, $A \subseteq B \cup C$, $A \supseteq B \cup C$, $A \subseteq B \cap C$, $A \supseteq B \cap C$, for given sets A, B, C.
- prove using induction
- prove or disprove that a relation is an equivalence relation. Know the definitions of equivalence classes and partitions.

Less important but recommended:

- Know how paradoxes arise in set theory and how to avoid them using axioms.
- Know how to define natural numbers using sets.

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1. Prove or disprove:
   (a) The set $\mathbb{R}^2$ is the union of all lines through the origin $(0,0)$.
   (b) The set $\mathbb{R}^2$ is the union of all lines of the form $\{(x,y) \in \mathbb{R}^2 : ax + by = 0\}$ where a and b are rational numbers (i.e. all lines with rational slopes, together with the vertical line).
2. A set $A \subseteq \mathbb{R}$ is called *well-ordered* if every non-empty subset $X \subseteq A$ contains a smallest element in X.
   (a) Prove that if A is well-ordered, then any subset $B \subseteq A$ is well-ordered.
   (b) Prove that every finite subset of $\mathbb{R}$ is well-ordered, using induction on the number of elements.
3. Prove by induction and from the definitions that every natural number is either even or odd.
4. Are the following relations equivalence relations on $\mathbb{R}$?
   (a) xRy if there exist $a, b \in \mathbb{Q}$ such that $x - y = a + b\sqrt{2}$.
   (b) xRy if there exist $a, b \in \mathbb{Q}$ such that $x + y = a + b\sqrt{2}$.
5. Let R be a relation on $\mathcal{P}(\mathbb{R})$ defined as follows: for any subsets $A, B \subseteq \mathbb{R}$,
   $$ARB \text{ if there exists a real number } c \text{ such that } A = \{x + c : x \in B\}.$$
   Is R an equivalence relation on $\mathcal{P}(\mathbb{R})$?
6. For a natural number $n \in \mathbb{N}$ (using the set theory definition) we define $n + 1$ to be its successor n^+. Give a definition of $n + 2$ and $n + 3$. Show that $2 + 3 = 3 + 2$.