Cardinalities

1. Find a bijection between a unit disk with boundary and a unit disk without boundary.
2. Find a bijection between a square (with or without boundary) and a line segment (with or without end points).
3. A complex (or real) number is called *algebraic* if it satisfies a polynomial equation with rational coefficients (e.g. $\sqrt{2}$ is algebraic because it satisfies $x^2 - 2 = 0$). Prove that non-algebraic numbers exist. They are called *transcendental*.
4. A complex (or real) number is called *computable* if there is a computer program that can compute it to arbitrary precision. The precision (the number of digits after the decimal point) is taken as an input. Prove that non-computable numbers exist.
5. We will call a real (or complex) number *describable* if it can be unambiguously described using a finite string of letters over a finite alphabet, say, English. For example, π can be described as "the ratio of the circumference to the diameter of a circle". Prove that non-describable numbers exist.
6. Explain why there is no well-ordering of $\mathbb{R}$ that can be explicitly described using a finite alphabet.
7. Prove that for any set A, we have $|A| < |\mathcal{P}(A)|$, that is, there is an injection but no surjection from A to $\mathcal{P}(A)$. (Hint: this is very similar to Cantor's diagonal argument.)
8. Rank the following sets according to their cardinalities:
$$\mathbb{N},\quad \mathbb{N}^2,\quad \mathbb{R},\quad \mathbb{R}^2,\quad \mathcal{P}(\mathbb{N}),\quad \mathcal{P}(\mathbb{R}),\quad \{0,1\}^{\mathbb{N}},\quad \{0,1\}^{\mathbb{R}},\quad \mathbb{N}^{\{0,1\}},\quad \mathbb{N}^{\mathbb{R}},\quad \mathbb{R}^{\mathbb{N}}$$

Useful facts.

1. Any subset of a countable set is countable.
2. Any countably infinite set has the same cardinality as $\mathbb{N}$.
3. The Cartesian product of finitely many countable sets is countable.
4. The union of countably many countable sets is countable.
5. For every non-negative real number α and every integer $b \geq 2$, we can find a base-b expansion of α, i.e.
$$\alpha = n + \sum_{i\geq 0} a_i/b^i$$
where n is an integer and a_i is an integer between 0 and $b-1$. This is denoted $\alpha = n.a_1a_2a_3a_4\ldots$. We can deal with negative numbers α by first finding an expansion of $|\alpha|$.
6. The real number α is rational if and only if its base b expansion either terminates for repeats (e.g. $1/7 = 0.142857142857\ldots$ in base 10 but it is equal to 0.1 in base 7.).
7. The base b expansion is unique unless it ends in all 0's or all $(b-1)$'s. (So, it is unique if we never allow it to end in all $(b-1)$'s.)