Name:
Collaborators:
Outside resources:

Math 2106, Foundations of Mathematical Proof
HW 7 — Due April 21 (Friday)

Reference: *Abstract Algebra* by Thomas Judson. http://abstract.pugetsound.edu

1. Prove that composition of functions is associative. That is, for any functions $f : A \to B$, $g : B \to C$, and $h : C \to D$ show that $h \circ (g \circ f) = (h \circ g) \circ f$.

2. Let $S = \mathbb{R} - \{-1\}$ and define a binary operation on S by $a * b = a + b + ab$. Prove that $(S, *)$ is an abelian group.

3. Prove that the set
$$S = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$
form a non-abelian group under matrix multiplication, called the *Heisenberg group*.

4. Write out the Cayley table for the group $(\mathbb{Z}_4, +)$.

5. Let $U(8) = \{[1], [3], [5], [7]\}$ where $[i]$ denotes the equivalence class of i modulo 8. Write out the multiplication table for $U(8)$ under the operation $[a] \cdot [b] \mapsto [ab]$. Prove that $U(8)$ is a group under this operation. Is $U(8)$ the same as $\mathbb{Z}_4$ up to relabeling?
(The group $U(8)$ is called $\mathbb{Z}_8^*$ in some textbooks.)

6. Write out the Cayley table for the group G formed by the symmetries of a rectangle that is not a square. Is G the same as either $\mathbb{Z}_4$ or $U(8)$ up to relabeling?

7. Show that up to relabeling there are exactly two different groups of order 4.
(One is called the *cyclic group of order* 4 and the other is called the *Klein four-group*.)

8. Let G be a finite group with identity $e \in G$.

   (a) Prove that for each $g \in G$, there exists an integer $n > 0$ such that $g^n = e$.

   (b) Prove that there exists an integer $m > 0$ such that $g^m = e$ for all $g \in G$.

   Note: $g^k = \underbrace{g * \cdots * g}_{k \text{ times}}$.

9. Prove that the inverse of $g_1 g_2 \cdots g_n$ in a group is $g_n^{-1} \cdots g_2^{-1} g_1^{-1}$.

10. Prove that if $g^2 = e$ for all $g \in G$, then G is abelian.

11. Prove that a group G is abelian if and only if $(gh)^2 = g^2 h^2$ for all $g, h \in G$.

12. Prove that a group G is abelian if and only if $(gh)^{-1} = g^{-1}h^{-1}$ for all $g, h \in G$.

13. (a) Give an example of an infinite group G and an infinite subgroup $H \subsetneq G$ such that $[G : H]$ is finite.

    (b) Give an example of an infinite group G and an infinite subgroup $H \subsetneq G$ such that $[G : H]$ is infinite.

14. Let G be a group and H be a subgroup. Suppose $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$. Show that left cosets and right cosets coincide, that is, $gH = Hg$ for all $g \in G$.