Final review problems

1. Write two implications that are true but whose converses are false.
2. Prove that there are infinitely many natural numbers n such that $\sqrt{n}$ is irrational.
3. Prove that there does not exist natural numbers x and y such that $x^2 - y^2 = 1$.
4. Consider the relation R in $\mathbb{Z}_5$ defined by aRb if $a^2 \equiv b^2 \pmod 5$. Prove that R is an equivalence relation and describe the equivalence classes.
5. Prove or disprove: Let A be a set.
   (a) The intersection of two equivalence relations on A is again an equivalence relation on A.
   (b) The union of two equivalence relations on A is again an equivalence relation on A.
6. (a) Give an example of two sets A and B such that $|B^A| = 81$.
   (b) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is injective but not surjective.
   (c) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is surjective but not injective.
   (d) Prove that if a function f is injective, then $f(C \cap D) = f(C) \cap f(D)$ for any subsets C, D of the domain.
   (e) Give a concrete example to show that the statement above is false when f is not injective.
7. Determine whether the following sets are countable. Prove your assertions.
   (a) The set of sequences of integers.
   (b) The set of sequences of integers in which all but finitely many entries are zero.
8. Prove or disprove:
   (a) Convergent sequences are bounded.
   (b) Bounded sequences are convergent.
9. Suppose a sequence s_n converges to a positive real number L.
   (a) Prove that there exists an integer M such that $s_n > 0$ for all $n \geq M$.
   (b) Prove that the sequence $\sqrt{|s_n|}$ converges to $\sqrt{L}$.
10. Let G be a finite group (where composition is written multiplicatively). Let $g \in G$.
    (a) Prove that the set $\langle g \rangle = \{g^n : n \in \mathbb{N}\}$ forms a subgroup of G.
    (b) Prove that if G has prime order and $g \neq e$, then $\langle g \rangle = G$.
11. (a) Prove that groups of prime order are abelian.
    (b) Prove that up to relabeling there is exactly one group of each prime order.
12. List all groups of order ≤ 7 up to relabeling.