Prove that the linear-fractional transformations mapping the disk $|z| < 1$ onto itself are those of the form

$$\psi(z) = \frac{\lambda(z - z_0)}{\overline{z_0}z - 1}$$

where $|z_0| < 1$ and $|\lambda| = 1$.

Solution: Note that this problem has two parts: we must show that LFTs of the above form map the disk onto itself AND that every LFT mapping the disk onto itself is of this form.

For the first part, observe that, given ψ of the above form,

$$\left|\psi\left(e^{i\theta}\right)\right| = |\lambda| \left|\frac{e^{i\theta} - z_0}{\overline{z_0}e^{i\theta} - 1}\right| = \left|\frac{e^{i\theta/2} - e^{-i\theta/2}z_0}{e^{i\theta/2}\overline{z_0} - e^{-i\theta/2}}\right| = \frac{\left|e^{i\theta/2} - e^{-i\theta/2}z_0\right|}{\left|\overline{e^{i\theta/2} - e^{-i\theta/2}z_0}\right|} = 1$$

for any $\theta \in [0, 2\pi)$. Thus ψ maps the unit circle into the unit circle. Since ψ is a LFT, we conclude that it maps the unit circle ONTO the unit circle (since the image of the unit circle must be a line or a circle). Since ψ is a LFT that maps the unit circle to the unit circle, it must map the interior of the disk either onto the interior of the disk or the exterior of the disk. Since

$$\psi(0) = \lambda z_0,$$

we conclude that the interior of the disk is mapped onto the interior of the disk.

In the opposite direction it is useful first to consider LFTs mapping the disk onto the disk and fixing 0. A LFT that fixes zero must be of the form

$$\psi(z) = \frac{az}{bz + c},$$

with a and c non-zero. Suppose that ψ also maps the disk onto itself. Then it must map the unit circle $|z| = 1$ onto itself, so we must have

$$\left|\frac{ae^{i\theta}}{be^{i\theta} + c}\right| = 1$$

for all $\theta \in [0, 2\pi)$. Thus

$$1 = \left|\frac{b}{a} + \frac{c}{a}e^{-i\theta}\right|^2 = \left|\frac{b}{a}\right|^2 + 2\frac{\operatorname{Re}\left(\overline{b}ce^{-i\theta}\right)}{|a|^2} + \left|\frac{c}{a}\right|^2.$$

This is possible, for all $\theta \in [0, 2\pi)$, only if $\overline{b}c = 0$. Since $c \neq 0$, we conclude that $b = 0$. Hence $\psi(z) = \lambda z$ with $\lambda = {}^{c}\!/_{a}$. Since $1 = |\psi(1)| = |\lambda|$, we see that ψ is of the claimed form, with $z_0 = 0$.

For a general LFT $\psi(z)$ mapping the disk onto the disk, let $z_0 = \psi(0)$ and let

$$\phi(z) = \frac{z - z_0}{\overline{z_0}z - 1}.$$

By the first part of the proof, ϕ is a LFT that maps the disk onto itself. Thus

$$\phi \circ \psi(z) = \frac{\psi(z) - z_0}{\overline{z_0}\psi(z) - 1}$$

is a LFT that maps the disk onto itself. Furthermore $\phi \circ \psi(0) = 0$. Hence, by what we just proved, $\phi \circ \psi(z) = \lambda z$ with $|\lambda| = 1$, i.e.,

$$\frac{\psi(z) - z_0}{\overline{z_0}\psi(z) - 1} = \lambda z.$$

Solving for $\psi(z)$ yields

$$\psi(z) = \frac{\lambda z - z_0}{\lambda \overline{z_0} z - 1} = \frac{\lambda\,(z - w_0)}{\overline{w_0} z - 1},$$

with $w_0 = \overline{\lambda} z_0$. Thus ψ is of the claimed form.