COMPLEX ANALYSIS

1) Find a real harmonic function h defined on $\mathbb{C} \setminus \{0\}$, which has no harmonic conjugate in $\mathbb{C} \setminus \{0\}$. Justify your example.

2) Let f be an entire function. Let $k \in \mathbb{N}$ and $C, R_0 \in (0, \infty)$. Suppose that

$$|f(z)| \leq C|z|^k,$$

for any $z \in \mathbb{C}$ with $|z| \geq R_0$. Prove that f is a polynomial of degree $\leq k$.

3) Let f be a holomorphic function on a simply connected plane domain U. Suppose that $f(z) \neq 0$ for all $z \in U$. Prove that there exists a holomorphic function g on U such that $f(z) = e^{g(z)}$ for all $z \in U$.

4) Let D be a plane domain such that $\overline{z} \in D$ whenever $z \in D$. Let f be an analytic function defined on D. Suppose that $f(z) \in \mathbb{R}$ for any $z \in D \cap \mathbb{R}$. Prove that $f(\overline{z}) = \overline{f(\overline{z})}$ for any $z \in D$.

5) Let $\{f_n\}_{n \in \mathbb{N}}$ be a normal family of analytic functions defined on a plane domain U. Suppose that for each $z \in U$, the sequence $\{f_n(z)\}$ converges. Show that $\{f_n\}$ converges uniformly on every compact subset of U.