Qualifying Exam Complex Analysis August, 2015

1. Let f be a holomorphic function defined on $\mathbb{C}$. Suppose that $\lim_{|z|\to\infty} |f(z)| = \infty$. Prove that f is a complex polynomial.

2. Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Answer the following questions with a proof or example.

   (a) Is there a holomorphic function $f : \mathbb{C} \to \mathbb{H}$ such that $f(\mathbb{C}) = \mathbb{H}$?

   (b) Is there a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that $f(\mathbb{H}) = \mathbb{C}$?

   Note that we do not require that f is injective.

3. Compute the definite integral
$$\int_0^\infty \frac{\sin^2 x}{x^2} dx.$$

4. Let h be a real harmonic function defined on the annulus $A = \{z \in \mathbb{C} : a < |z| < b\}$, where $0 < a < b < \infty$. Prove that there is $C \in \mathbb{R}$ such that $h - C \log |z|$ has a harmonic conjugate in A. Hint: Consider the function $\frac{\partial h}{\partial x} - i \frac{\partial h}{\partial y}$ and use Laurent series expansion.

5. Let D be a complex domain, and $(f_n)_{n\in\mathbb{N}}$ be a sequence of injective analytic functions defined on D. Suppose that f_n converges to f uniformly on every compact subset of D. Prove that f is either constant or injective on D. Hint: Apply Rouché's Theorem.