# MATH 105 211 INTEGRAL CALCULUS HW4

Due date: February 10th. Hand in Q2, Q10, and any two from Q4-Q9.

**Riemann Sum** Recall that if $f$ is an integrable function on $[a,b]$, one can approximate the integral $\int_a^b f(x)dx$ by a (limit of) Riemann sum:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \left( f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x \right),$$

where $\Delta x = \frac{b-a}{n}$ and each $x_i^*$ is a point in $[a+(i-1)\Delta x, a+i\Delta x]$. The above can be useful when evaluating some limit of infinite sums.

**Example 0.1.** Evaluate the following limit:

$$\lim_{n\to\infty} \left( \frac{1}{1+\frac{3}{n}} \cdot \frac{3}{n} + \frac{1}{1+\frac{6}{n}} \cdot \frac{3}{n} + \cdots + \frac{1}{1+\frac{3n}{n}} \cdot \frac{3}{n} \right)$$

Since all the term involves $\frac{3}{n}$, it is natural to guess $\Delta x = \frac{3}{n} = \frac{b-a}{n}$ (we do not know what are $a, b$: Just guessing for the moment). Since the term (besides $\frac{3}{n}$) in the limit looks like

$$\frac{1}{1+\Delta x}, \frac{1}{1+2\Delta x}, \cdots \frac{1}{1+n\Delta x} = \frac{1}{1+3},$$

we observe that $a = 1$, $b = 4$ and $f(x) = \frac{1}{x}$. Indeed

$$\lim_{n\to\infty} \left( \frac{1}{1+\frac{3}{n}} \cdot \frac{3}{n} + \frac{1}{1+\frac{6}{n}} \cdot \frac{3}{n} + \cdots + \frac{1}{1+\frac{3n}{n}} \cdot \frac{3}{n} \right) = \int_1^4 \frac{1}{x}dx = \ln|x| \Big|_1^4 = \ln 4,$$

where we use of course the Fundamental Theorem of Calculus in the last step. Note that the limit is the limit of right hand Riemann sum of the function $\frac{1}{x}$ on $[1,4]$.

**Example 0.2.** Note that the choice of $a, b$ and $\Delta x$ can be quite arbitrary. Indeed, one can always set $a = 0$, $b = 1$ and $\Delta x = \frac{1}{n}$ (but then the integrand $f$ would be different). Take again the limit in example 1. The point is to (i) isolate the term $\frac{1}{n}$ and (ii) look up the term $\frac{i}{n}$ and change that into $x$. Now the general term is like

$$\frac{1}{1+\frac{3i}{n}} \cdot \frac{3}{n} = \frac{3}{1+3\frac{i}{n}} \cdot \frac{1}{n},$$

so we set $f(x) = \frac{3}{1+3x}$ and the limit is equal to

$$\int_0^1 \frac{3}{1+3x}dx.$$

Q1: Evaluate $\int_0^1 \frac{3}{1+3x}dx$.

Q2: Evaluate

$$\lim_{n\to\infty} \left( \frac{(2+4/n)^2}{5n} + \frac{(2+8/n)^2}{5n} + \cdots + \frac{(2+4n/n)^2}{5n} \right)$$

Q3: Evaluate

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^n k \left( \frac{k^2}{n^2} + 1 \right)^{10}.$$

**Substitution: Evaluate the following integrals**

Q4: $\int (x+3)(x-1)^5 dx$.

Q5: $\int \frac{3}{x \ln x} dx$.

Q6: $\int \sqrt{4-\sqrt{x}} dx$.

Q7 $\int x^3 \sqrt{1-x^2} dx$.

Q8 $\int e^{x+e^x} dx$.

Q9 $\int \frac{x}{1-x^2+\sqrt{1-x^2}} dx$.

Q10 Show that for any integrable function $f$,

$$\int_0^1 f(\sqrt{x}) dx = \int_0^1 x f(x) dx.$$

Q11 If $f'(1) = 2$ and $f'(2) = 4$, find $\int_1^2 f'(x) f''(x) dx$.