INDIAN INSTITUTE OF TECHNOLOGY ROORKEE

CSN-523: Computational Geometry

Lecture 21: Polygon Partitioning, Convex Partitioning

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Piazza Class Room: https://piazza.com/iitr.ac.in/spring2017/csn523/home

Moodle Site: http://moodle.iitr.ac.in/course/view.php?id=23 [Enrollment Key: csn523@2017]

Polygon Triangulation:

Summarizing

Running time of polygon triangulation:

- $O(n^2)$ by substracting ears
- $O(n^2)$ by inserting diagonals
- $O(n \log n)$ by:
  1. Decomposing the polygon into monotone subpolygons in $O(n \log n)$ time
  2. Triangulating each monotone subpolygon in $O(n)$ time

Is it possible to triangulate a polygon in $o(n \log n)$ time?

Yes.
There exists an algorithm to triangulate an n-gon in $O(n)$ time, but it is too complicated and, in practice, it is not used.

Convex Partitioning of a Polygon:

- A partition into triangles can be viewed as a special case of partition into convex polygons
- Two goals of partition (conflicting):
  - Into as few convex pieces as possible
  - Do as quickly as possible
- Two types of partition:
  - By diagonals (endpoints must be vertices)
  - By segments (endpoints are any point in the boundary)

Convex Partitioning of a Polygon:

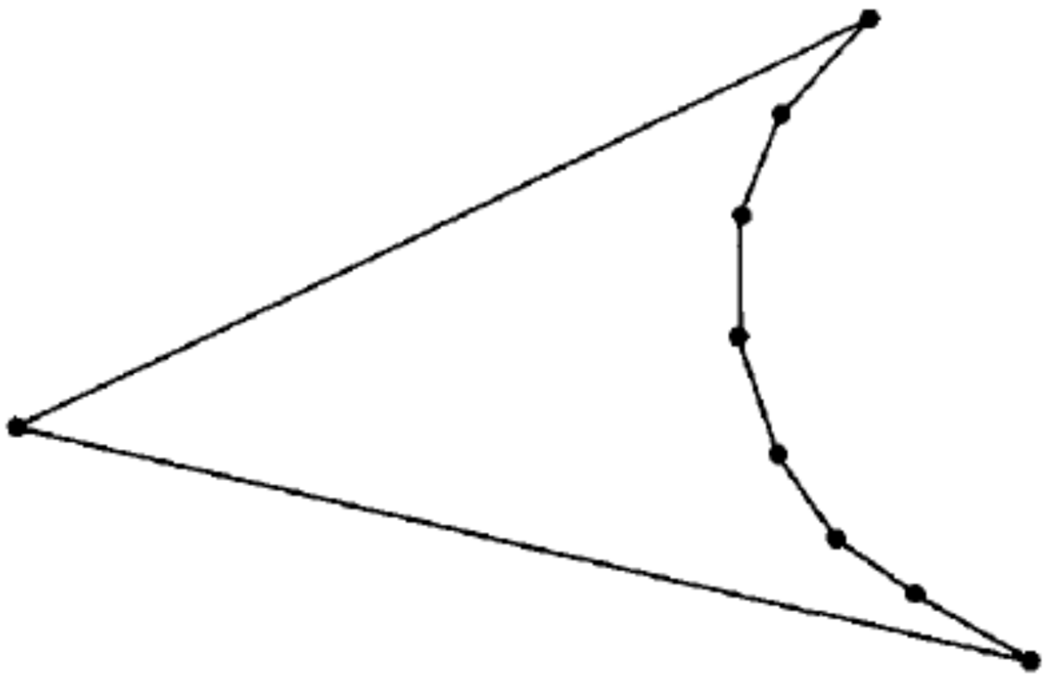


Fig. 1.26. Of a polygon's n vertices, as many as $n - 3$ may be reflex.

LEMMA 1.5 [Chazelle 1980]. Any polygon can be partitioned into at most $r + 1$ convex pieces.

Convex Partitioning of a Polygon:

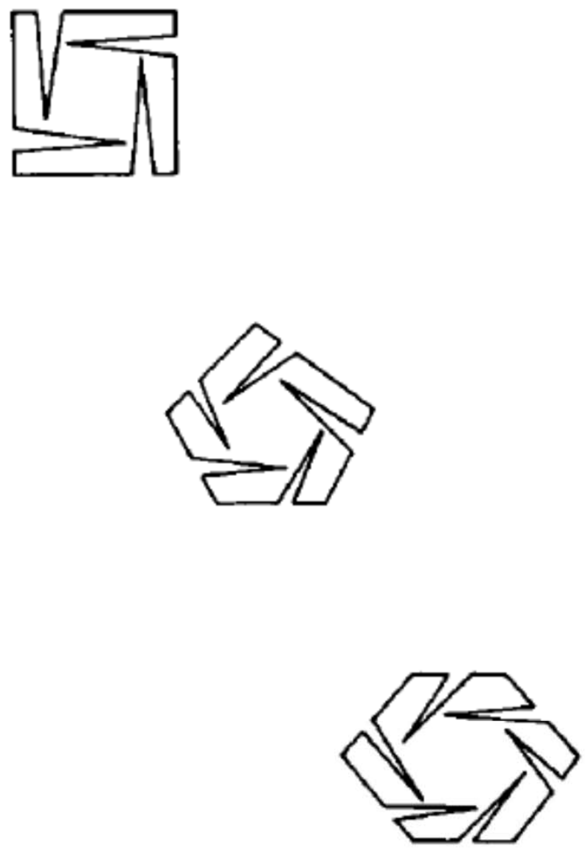


Fig. 1.27. "Shutter" shapes show that r guards can be necessary.

THEOREM 1.5 [O'Rourke 1982]. r guards are occasionally necessary and always sufficient to see the interior of a simple n-gon of $r \geq 1$ reflex vertices.

Convex Partitioning (O'Rourke):

- Competing Goals:
  - minimize number of convex pieces
  - minimize partitioning time
- Add (Steiner) points or just use diagonals and not add points?

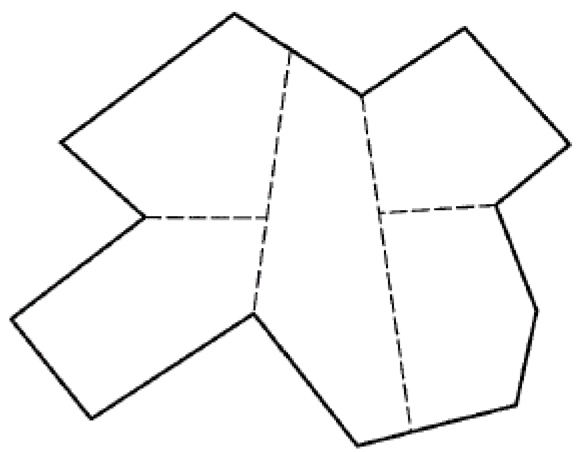
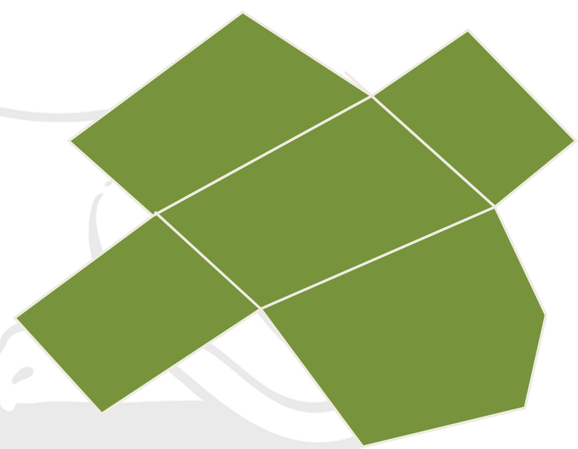


FIGURE 2.10 $r + 1$ convex pieces: $r = 4$; 5 pieces.

Adding segments with Steiner points.
r = number of reflex vertices



Adding only diagonals.

Bounds on Convex Partitions:

<u>Theorem</u> (Chazelle): Let Φ be the fewest number of convex pieces into which a polygon may be partitioned. For a polygon of *r* reflex vertices:

$$\left\lceil \frac{r}{2} \right\rceil + 1 \le \Phi \le r + 1$$

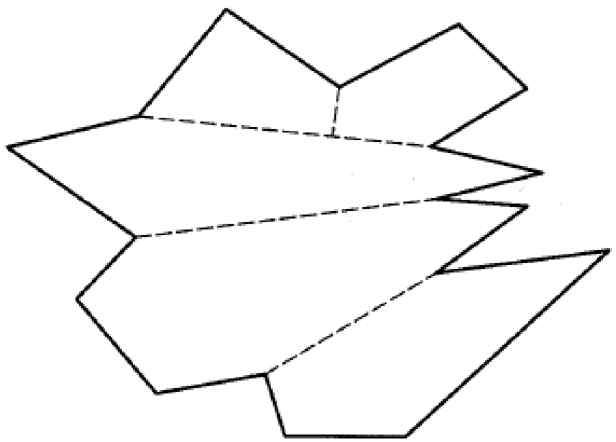


FIGURE 2.11 $\lceil r/2 \rceil + 1$ convex pieces: $r = 7$; 5 pieces.

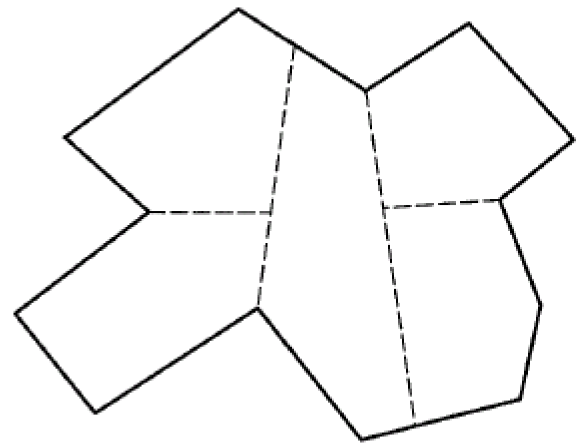


FIGURE 2.10 $r + 1$ convex pieces: $r = 4$; 5 pieces.

<u>Lower bound</u>:
Must eliminate all reflex vertices.
Single segment resolves at most 2 reflex angles.

<u>Upper bound</u>:
Bisect each reflex angle.

Convex Partitioning: Hertel and Mehlhorn Algorithm

- A very clean algorithm that partitions with diagonals quickly
  - has bounded "badness" in terms of the number of convex pieces
- A diagonal d *is* **essential** for vertex v if removal of d makes v non-convex
- The algorithm
  - start with a triangulation of P
  - remove an inessential diagonal
  - repeat

Convex Partitioning: Hertel and Mehlhorn Algorithm

Claim:

There can be at most two diagonals essential for each reflex vertex.

Corollary:

The algorithm is not worse than 4x optimal in the number of convex pieces.

Convex Partitioning: Hertel and Mehlhorn Algorithm

- *Essential* diagonal *d* for vertex *v* if removing *d* creates nonconvex piece at *v.*
- Start with any triangulation.
- Iteratively remove inessential diagonals.
- Can be done in O(n) time!

<u>Lemma 1</u>: There can be at most 2 diagonals essential for any reflex vertex.

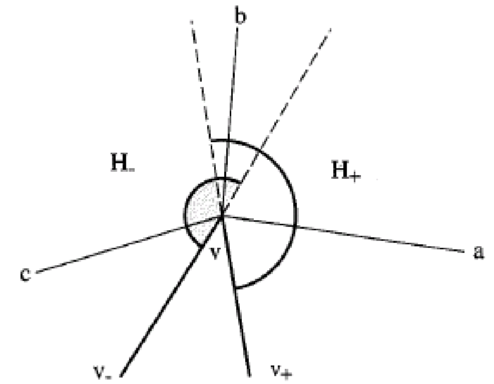


FIGURE 2.12 Essential diagonals. Diagonal a is not essential because b is also in H_+. Similarly c is not essential.

- By Lemma 1, number of essential diagonals $\leq 2r$
- Number of convex pieces: $\leq 2r + 1 < 2r + 4 \leq 4\Phi$
  - Recall: $\left\lceil \frac{r}{2} \right\rceil + 1 \leq \Phi \Rightarrow 4\left(\left\lceil \frac{r}{2} \right\rceil + 1\right) \leq 4\Phi \Rightarrow 4\left\lceil \frac{r}{2} \right\rceil + 4 \leq 4\Phi \Rightarrow 2r + 4 \leq 4\Phi$

Algorithms for Optimal Convex Partitioning of a Polygon (O'Rourke):

- Optimal convex partition using diagonals
  - Greene (1983): O(n^4) time with dynamic programming
  - Keil (1985): O(n^3log$n$) time with dynamic programming
- Optimal convex partition using arbitrary segments
  - Chazelle (1980) : O(n^3) time

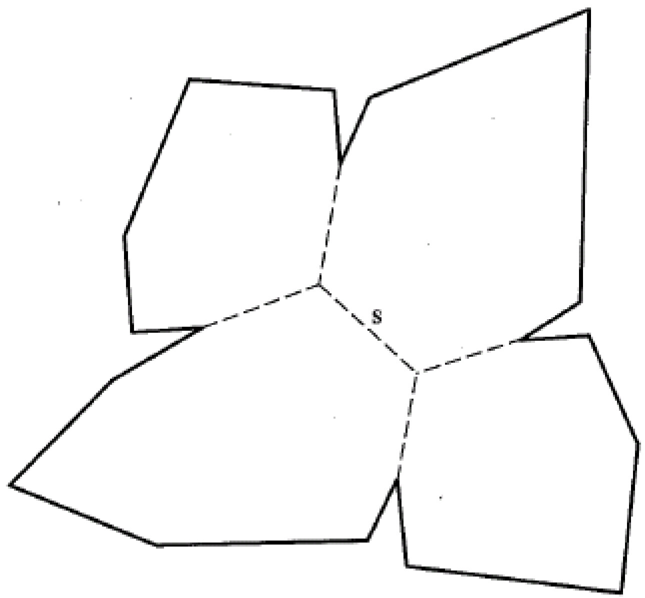


FIGURE 2.13 An optimal convex partition. Segment s does not touch ∂P.

CGAL Convex Polygon Partitioning:

- Y-Monotone partition:
  - de Berg *et al*.: $O(n \lg n)$ time (see earlier slides)
- Optimal convex partition using diagonals
  - Greene (1983): $O(n^4)$ time with dynamic programming
- Approximate convex partition removing inessential diagonals
  - Starts with triangulation:
    - 2D constrained triangulation
      - Run time depends on number of triangulation edges intersected by each polygon edge
  - Hertel/ Melhorn: $O(n)$ time after triangulation (see earlier slide) $\leq 4\Phi$
- Approximate convex partition using sweep-line $\leq 4\Phi$
  - Greene (1983): $O(n \lg n)$
  - Starts with y-monotone partition (de Berg *et al*.)

CSN-523: Computational Geometry

Lecture 21: Point Location, Proximity – Voronoi Diagrams

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Georgy Voronoy: (1868-1908)

- Georgy Feodosevich Voronoy was a Russian and Ukrainian mathematician. Among other things, he defined the Voronoi diagram.

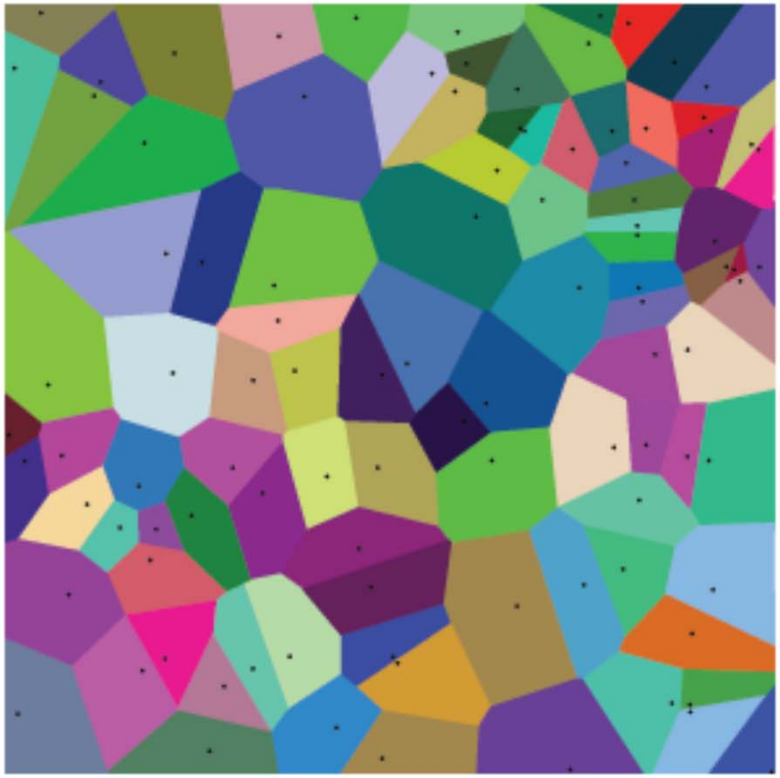
Boris Delaunay: (1890-1980)

- Boris Nikolaevich Delaunay or Delone was one of the first Russian mountain climbers and a Soviet/Russian mathematician.

- The triangulation is named after **Boris Delaunay** for his work on this topic from 1934.

Voronoi Diagram:

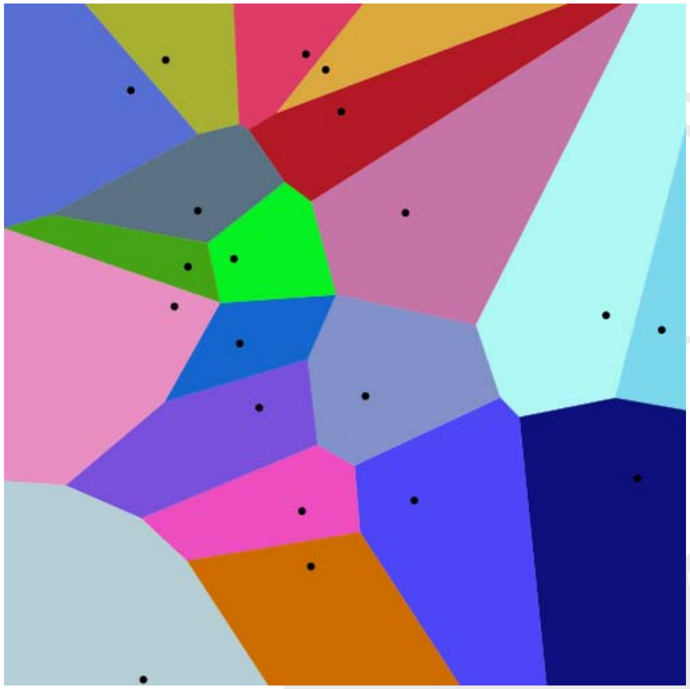
Example (What does it look like?)

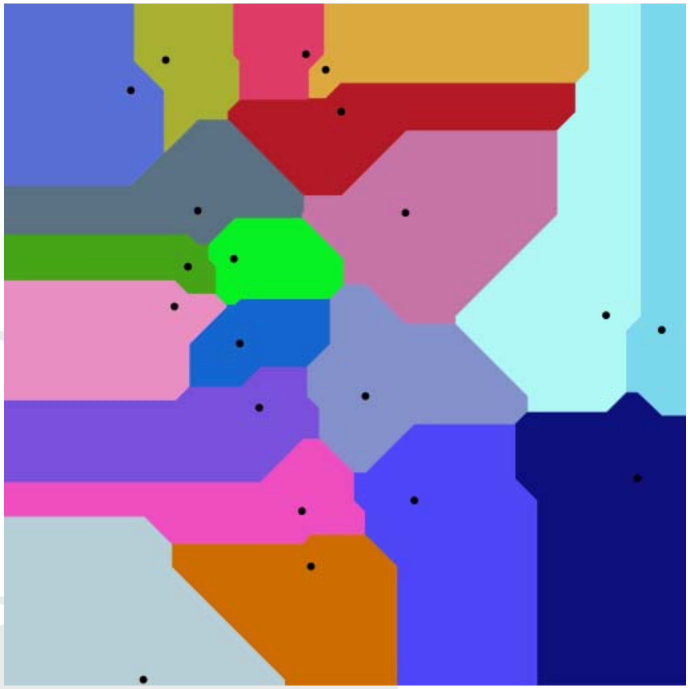


Each cell is a set of points whose nearest site is the same

Voronoi Diagrams of 2D point set: L_2 and L_1 distance norm

- Euclidean   Manhattan





Voronoi Diagram: History

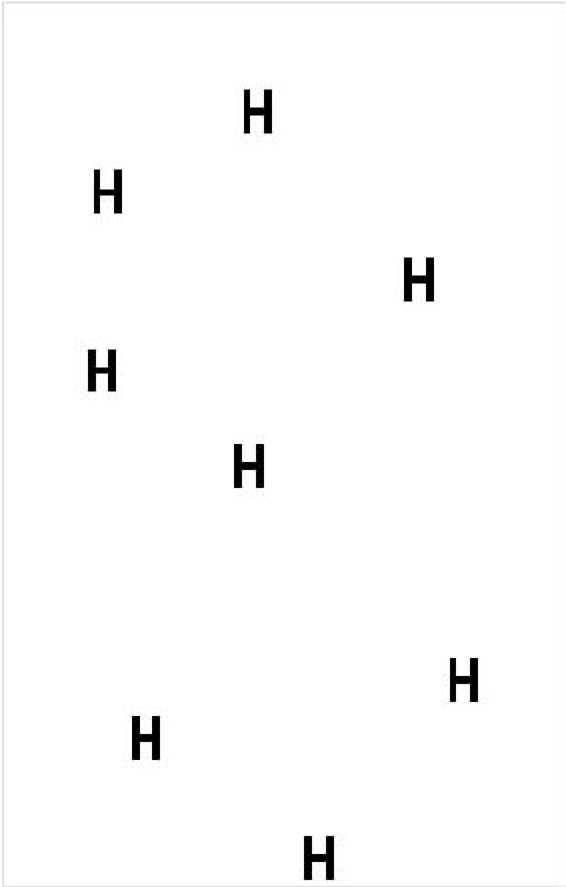
- The partitioning of a plane with points into convex polygons such that each polygon contains exactly one generating point and every point in a given polygon is closer to its generating point than to any other. A Voronoi diagram is sometimes also known as a Dirichlet tessellation. The cells are called Dirichlet regions or Voronoi polygons.

Voronoi Diagram: History

- Voronoi diagrams were considered as early at 1644 by René Descartes and were used by Dirichlet (1850) in the investigation of positive quadratic forms. They were also studied by Voronoi (1907), who extended the investigation of Voronoi diagrams to higher dimensions. They find widespread applications in areas such as computer graphics, epidemiology, geophysics, and meteorology. A particularly notable use of a Voronoi diagram was the analysis of the 1854 cholera epidemic in London, in which physician John Snow determined a strong correlation of deaths with proximity to a particular (and infected) water pump on Broad Street.

Voronoi Diagram: Applications

Given ambulance posts in a country, in case of an emergency somewhere, where should the ambulance come from?



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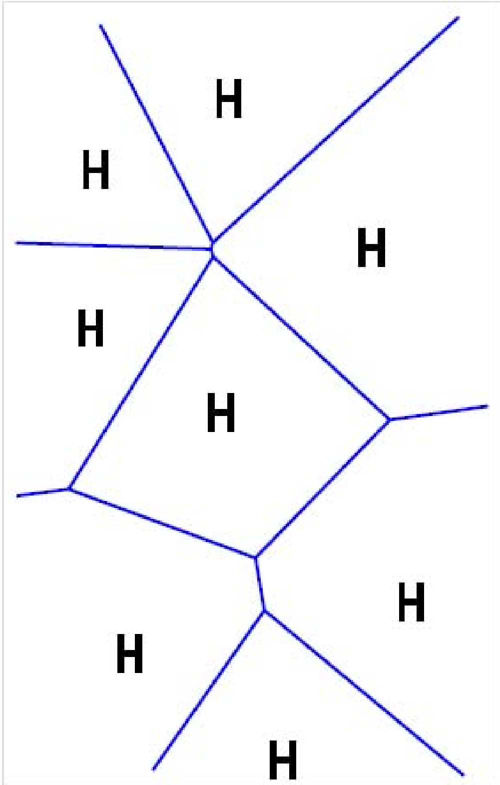
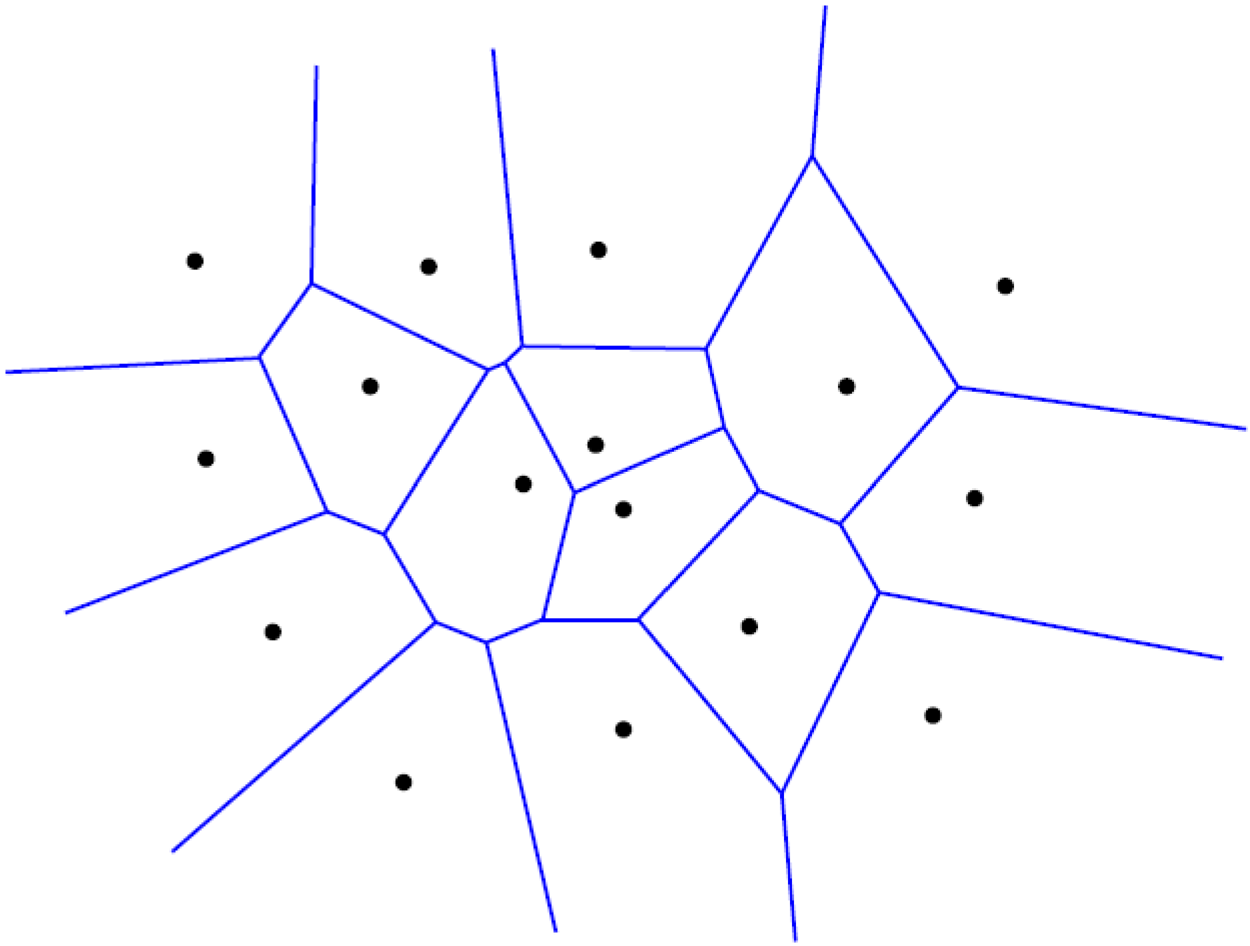


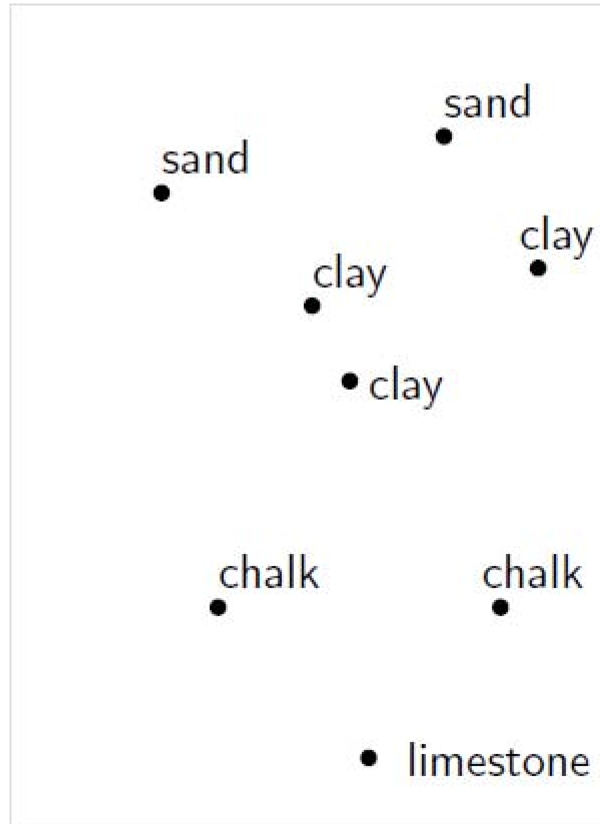
Voronoi diagram induced by a set of points (called sites): Subdivision of the plane where the faces correspond to the regions where one site is closest

Voronoi Diagram:



Voronoi Diagram: Spatial Interpolation

Suppose we tested the soil at a number of sample points and classified the results



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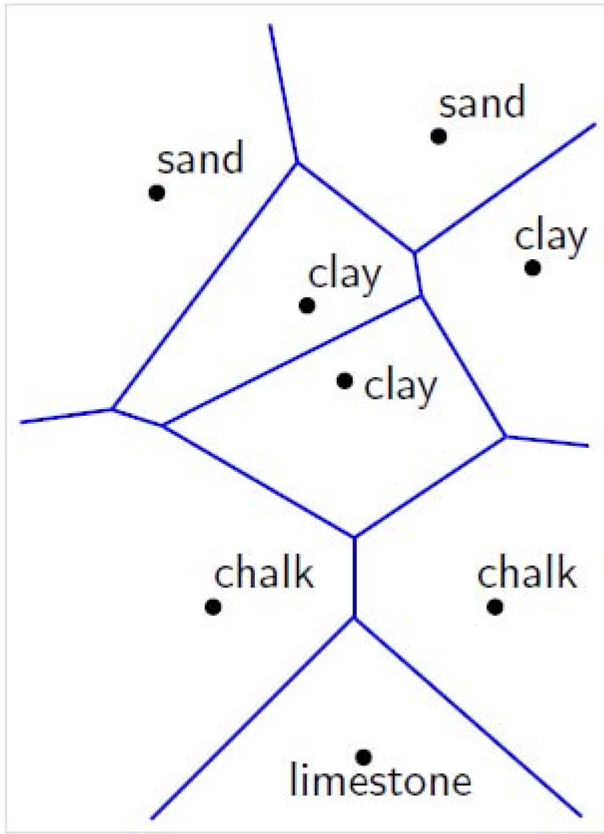

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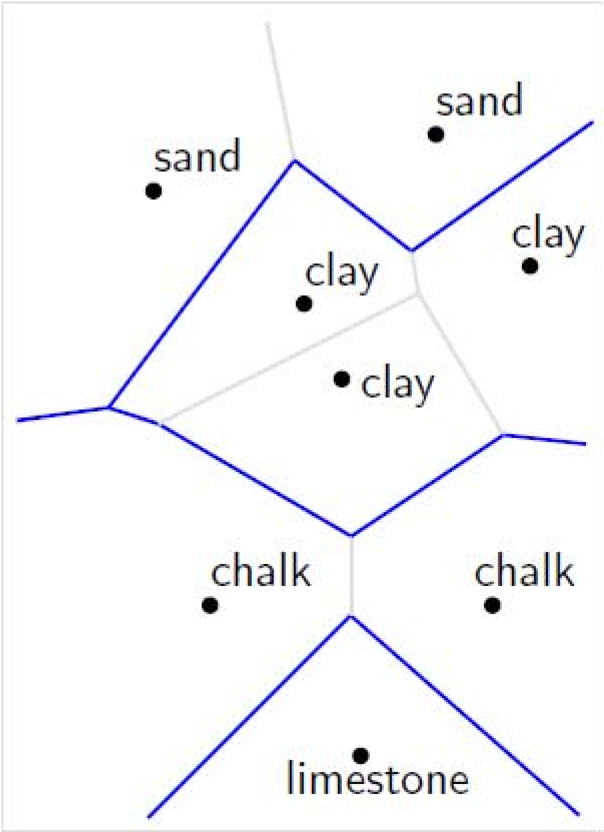
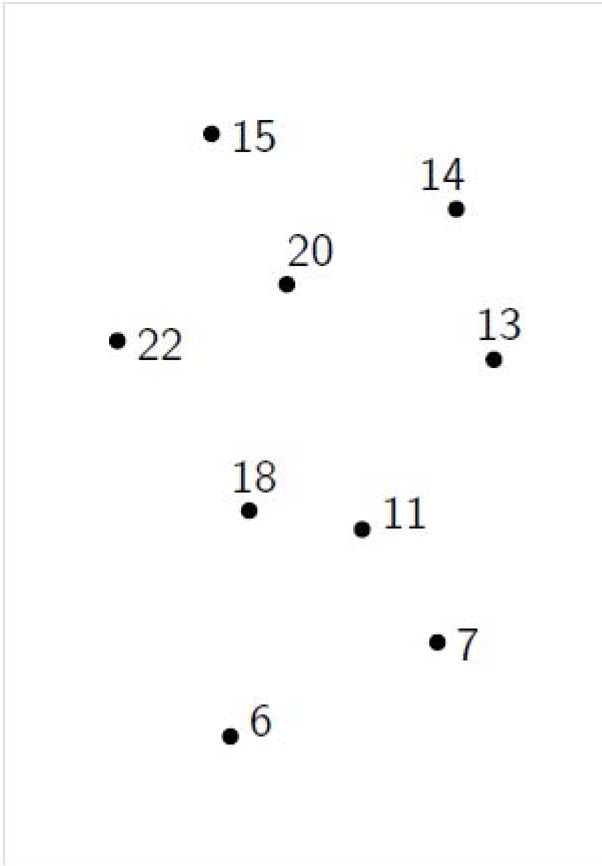


Voronoi Diagram: Spatial Interpolation

Suppose we measured the lead concentration at a number of sample points



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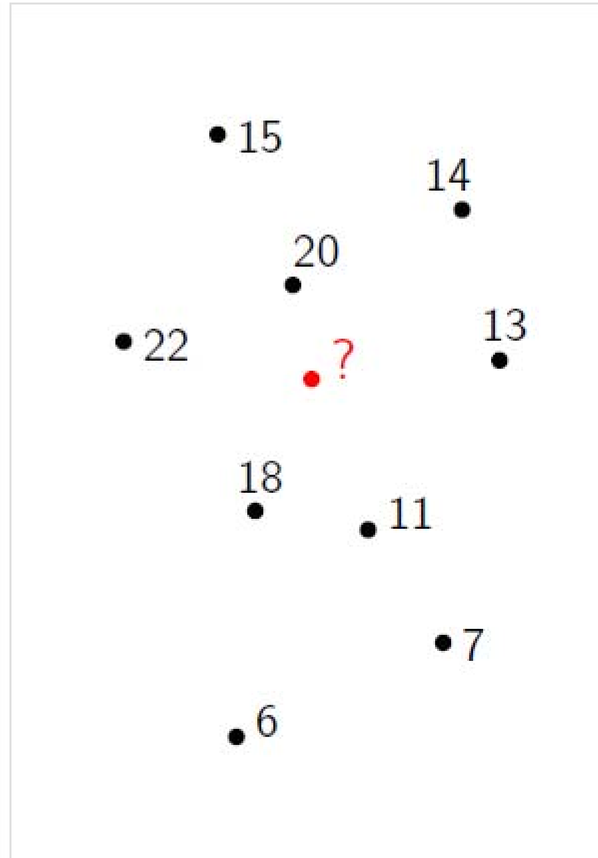

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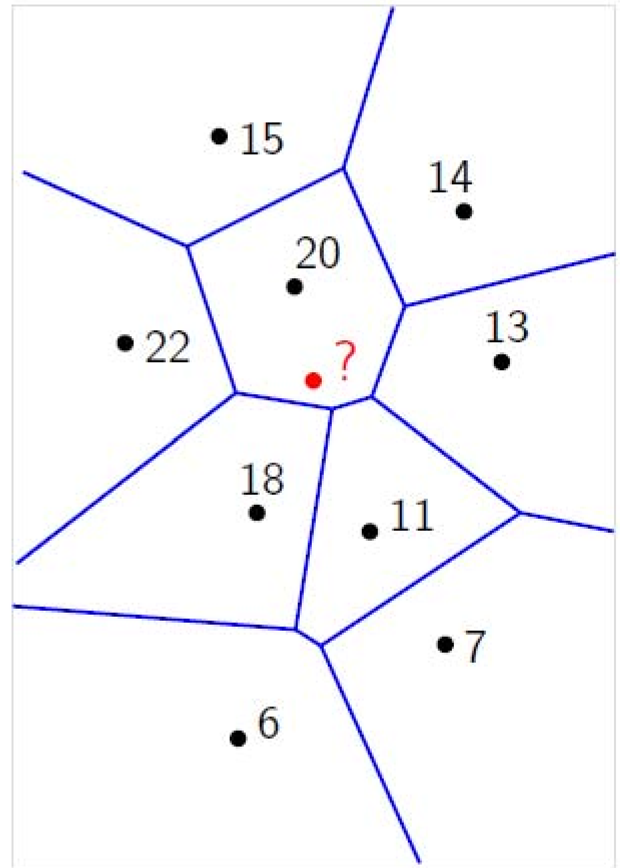

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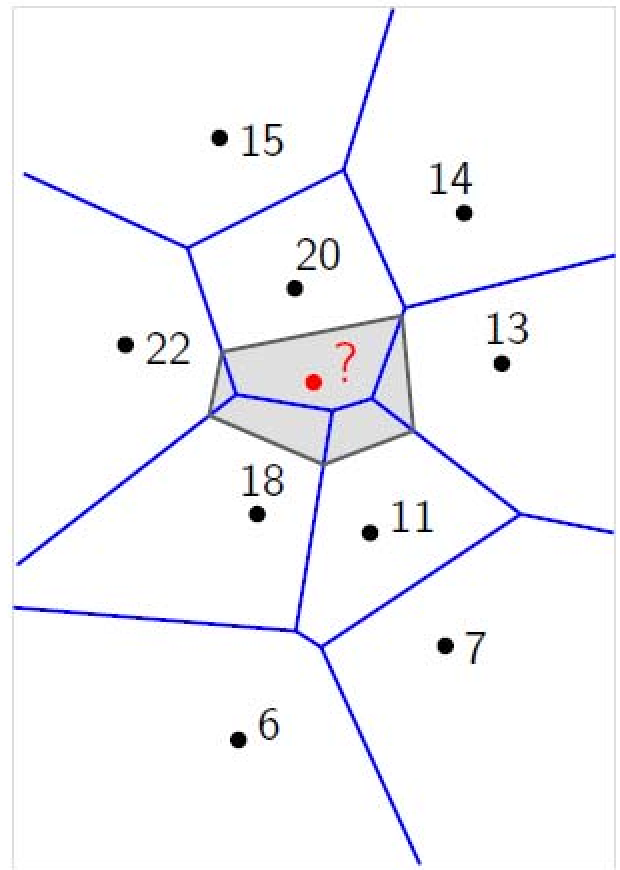

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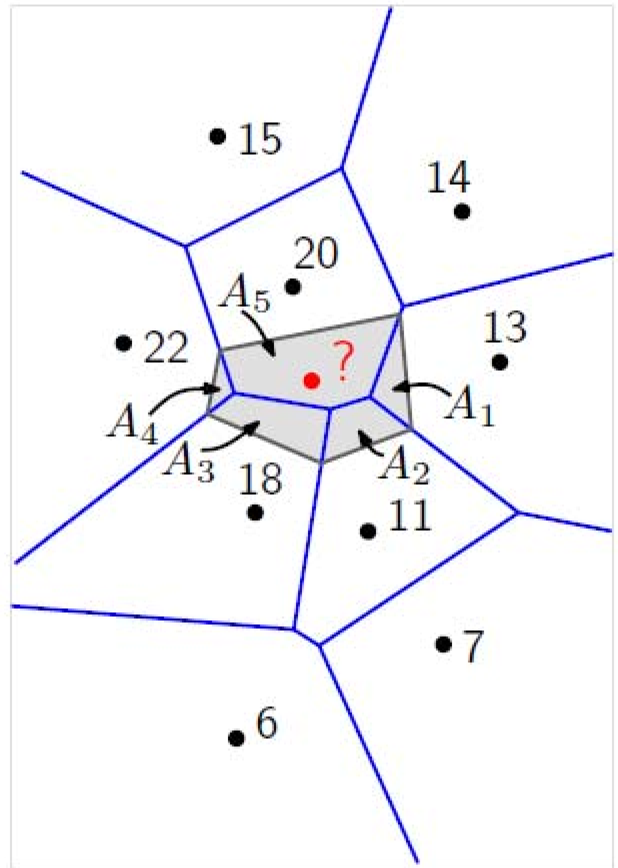

Voronoi Diagram: Spatial Interpolation

Let $A_T = A_1 + A_2 + \cdots + A_5$

The interpolated value is

$$\frac{A_1}{A_T} 13 + \frac{A_2}{A_T} 11 + \cdots + \frac{A_5}{A_T} 20$$