The splitting necklace problem

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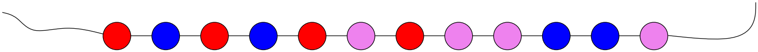
CERMICS, Optimisation et Systèmes

Two thieves and a necklace

n beads, t types of beads, a_i (even) beads of each type.

Two thieves: Alice and Bob.

Beads fixed on the string.

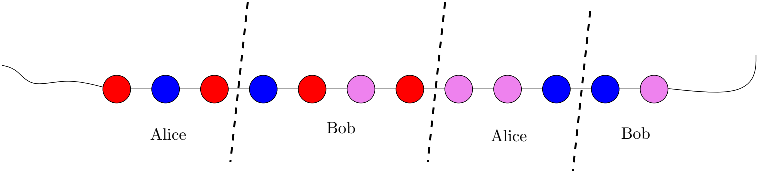


Fair splitting = each thief gets $a_i/2$ beads of type i

The splitting necklace theorem

Theorem (Alon, Goldberg, West, 1985-1986)

There is a fair splitting of the necklace with at most t cuts.



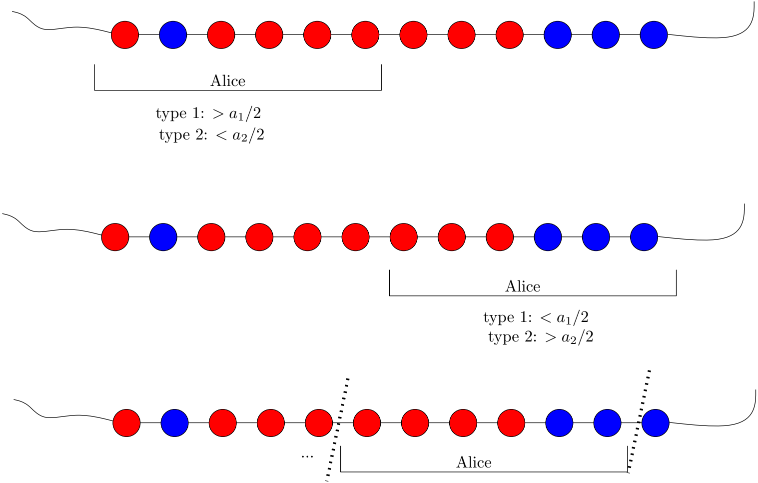
t is tight

t cuts are sometimes necessary:

Plan

1. Proofs and algorithms
2. A special case: the binary necklace problem
3. Generalizations
4. Open questions

Easy proof when there are two types of beads



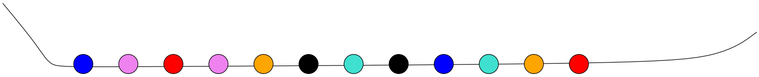
Definition

Binary Necklace Problem [Epping, Hochstättler, Oertel 2001]

Input. Necklace with t types of beads, 2 beads per type.

i.e. $n = 2t$ and $a_i = 2$ for all i.

Output. Fair splitting minimizing the number of cuts.



Defined in an operations research context as the paintshop problem (automotive industry)

Minimizing the number of cuts

Splitting necklace theorem: OPT $\leq t$, but obvious: greedy algorithm.

Challenge here: optimization.

Proposition (Epping, Hochstättler 2006)

The binary necklace problem is NP*-hard.*

Proof by MAX-CUT in 4-regular graphs.

M., Sebö 2009

- APX-hard

Gupta et al. 2013

- No polytime fixed-ratio approximation (assuming Unique Games Conjecture)

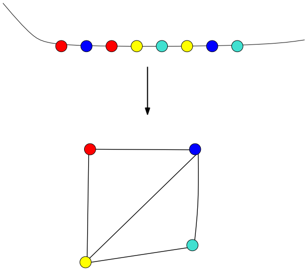
Positive results

M., Sebö 2009

OPT $\leq \frac{3}{4}t + \frac{1}{4}\beta$, *where* $\beta =$ *"forced cuts" (●● and ●●●). Can be found in polytime.*

Let $G = ([t], E)$, with $ij \in E$ if types i and j adjacent on necklace.

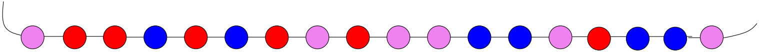
G *planar* $\Rightarrow$ *polytime.*



Generalizations

q thieves and a necklace

n beads, t types of beads, a_i (multiple of q) beads of each type.
q thieves: Alice, Bob, Charlie,...

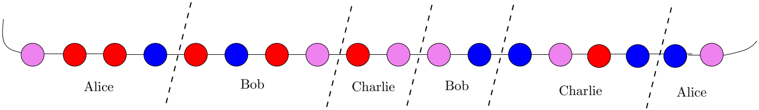


A generalization

Fair splitting = each thief gets a_i/q beads of type i

Theorem (Alon 1987)

There is a fair splitting of the necklace with at most $(q-1)t$ cuts.



$(q-1)t$ is tight

$(q-1)t$ cuts are sometimes necessary:

Open questions

- Complexity of computing a fair splitting with at most t cuts when there are two thieves.
- Complexity of computing a fair splitting with at most $(q-1)t$ cuts when there are q thieves.
- Existence of a fair splitting with choice of the advantaged thieves.
- Elementary proof of the splitting necklace theorem (any version).
- Simonyi's generalization for q thieves?

- Expected value of the optimum for the binary necklace problem.
- Extend results of the binary necklace problem to general necklaces (at least for two thieves).

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