University of Colorado **Boulder**

Department of Computer Science

CSCI 2824: Discrete Structures

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Lectures 9 and 10:
Sets and Set Operations

Announcements

- Homework 3 is posted. Due at the start of class on Friday.
- Quiz 1 (aka Mini-Exam 1) is this Wednesday
- Covers Hmwks 1-2 material (1.1-1.5 from Rosen). Anything covered in lecture slides or Hmwks 1-2 is fair-game
- You are allowed a standard 3in x 5in notecard of **handwritten** notes (and no magnifying glasses)

Sets and Set Operations

Def: A **set** is an unordered collection of objects, called elements or members of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that A is an element of set A. The notation $a \notin A$ denotes that a is not an element of the set A.

Notation: For sets with a small number of elements, we write the set with it's members inside braces: $\{a, b, c, d\}$

Example: The set V of all vowels: $V = \{a, e, i, o, u\}$

Example: The set of all primes less than 10: $P = \{2, 3, 5, 7\}$

Example: Thet set of all positive integers between 1 and 200:

$$A = \{1, 2, 3, \dots, 199, 200\}$$

Sets and Set Operations

Listing all elements is called the **roster method**. Equally popular is the **builder method**

Example: The set of all positive even integer less than 20

$$\{x \in \mathbb{Z}^+ \mid x \text{ is even and } x < 20\}$$

Note that this looks a lot like a quantified logic statement, but inside braces because it's a set

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ - Natural numbers
- $\mathbb{Z} = \{\ldots, -2 - 1, 0, 1, 2, \ldots\}$ - Integers
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ - Positive integers
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$
- $\mathbb{R} =$ the set of **real** numbers

Sets and Set Operations

Sets can have pretty much anything in them, even other sets

Example: $A = \{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$

Note: Sets have no ordering. So $\{1, 2, 3, 4\} = \{4, 2, 3, 1\}$

Note: It doesn't matter if there are repeated elements.

$$\{1, 2, 2, 2, 3, 3, 4\} = \{1, 2, 3, 4\}$$

Def: Two sets are equal if and only if they have the same elements.
So if A and B are sets, we say A and B are equal if and only if

$$\forall x \, (x \in A \leftrightarrow x \in B)$$

Sets and Set Operations

Def: The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of B.

Question: Use quantifiers to express the definition of $A \subseteq B$

$$\forall x \, [(x \in A) \rightarrow (x \in B)]$$

Finish the Sentence:

Example: The set of all integers, $\mathbb{Z}$, is a subset of ... $\mathbb{Q}$ or $\mathbb{R}$

Example: The set of all rational numbers, $\mathbb{Q}$, is a subset of ... $\mathbb{R}$

Example: The set of all 2824 students is a subset of ... All CU students

Sets and Set Operations

Strategy: To show that $A \subseteq B$ you have to show that every element of A is also in B

Strategy: To show that $A \nsubseteq B$ you have to find just one element in A that is not in B

Question: How do you feel about the following?

$$\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$$

When we wish to emphasize that $A \subseteq B$ but $A \neq B$ we write $A \subset B$. This says that A is a **proper** subset of B

Example: The set of all even integers is a proper subset of $\mathbb{Z}$

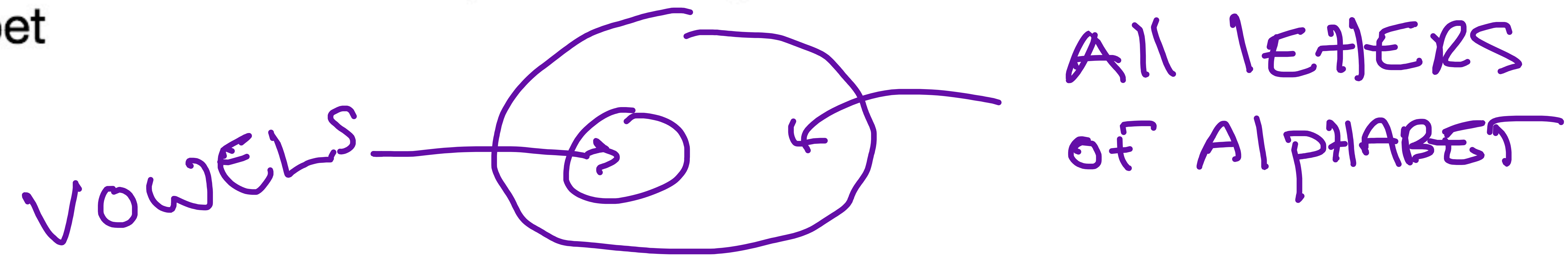
$$\{x \in \mathbb{Z} \mid x \text{ is even}\} \subset \mathbb{Z}$$

Sets and Set Operations

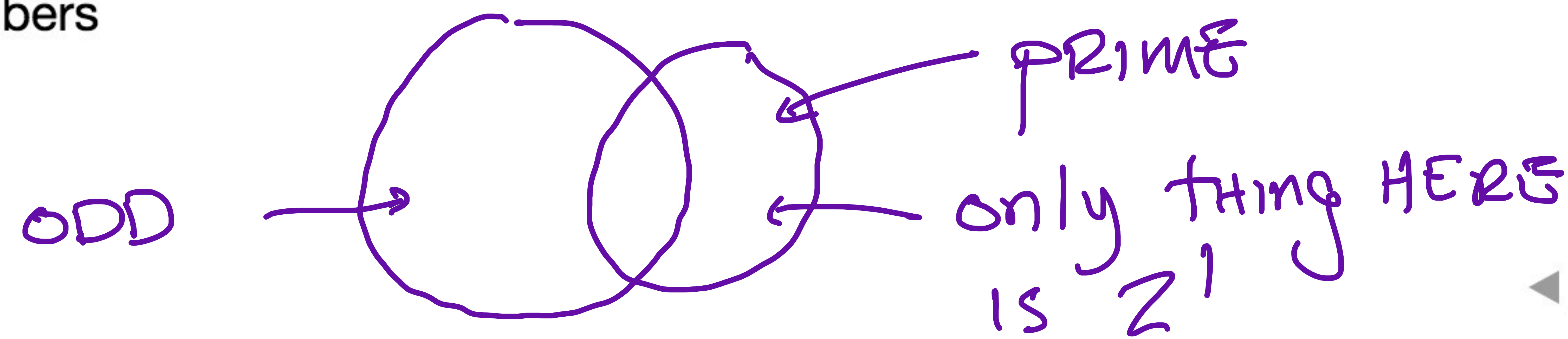
Venn Diagrams

When thinking about subsets, drawing a picture can help

Example: Draw a Venn Diagram relating vowels to the letters of the alphabet



Example: Draw a Venn Diagram relating prime numbers and odd numbers



Sets and Set Operations

Every nonempty set has at least two subsets

Theorem: For every set S, $\emptyset \subseteq S$, and $S \subseteq S$

Three Special Sets

The Empty Set: The set that has no elements, written $\emptyset$ or $\{\}$

The Singleton Set: A set with only one element,

$$\{a\}, \quad \{\text{Chris}\}, \quad \{\emptyset\}$$

The Power Set: $\mathcal{P}(S)$ set of all subsets of a set

Example: $\mathcal{P}(\{0,1,2\}) =$

$$\{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

Sets and Set Operations

Question: How many elements does $\mathcal{P}(S)$ contain?

$$\mathcal{P}(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

Answer: If S has n elements then $\mathcal{P}(S)$ has 2^n elements

The number of elements in a set is called the sets **cardinality**. If a sets cardinality is a finite number we say the set is **finite**:

Example: What is the cardinality of the english alphabet? $|A| = 2$

Example: What is the cardinality of the set of vowels? $|V| = 5$

Example: What is cardinality of the set of natural numbers?

THIS IS INFINITE. LATER WE'LL MAKE DISTINCTIONS B/W DIFFERENT SIZES OF ∞ CARDINALITY SETS!

Sets and Set Operations

Cartesian Products:

Sometimes we want to talk about things that come from sets, but have a defined order

Example: Think about ordered pairs of integers representing points in the $xy-$plane. $(3, 5)$ is distinctly different from $(5, 3)$.

We can generate all ordered pairs from two sets using the **Cartesian Product**;

Def: Let A and B be sets. The Cartesian product of A and B, written $A \times B$ is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence

$$A \times B = \{(a, b) \mid a \in A \wedge a \in B\}$$

Sets and Set Operations

Example: What is the Cartesian product of $A = \{Chris, Aly\}$ and $B = \{1, 2, 3\}$?

$A \times B = \{(Chris, 1), (Chris, 2), (Chris, 3), (Aly, 1), (Aly, 2), (Aly, 3)\}$

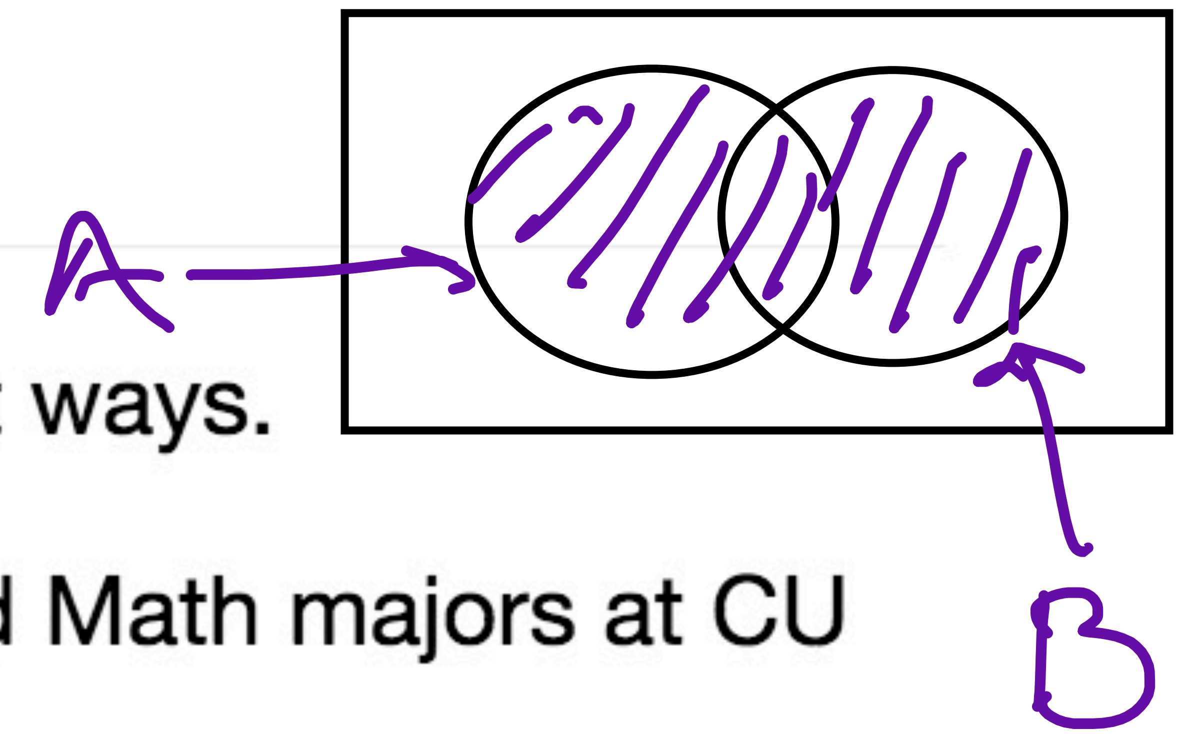
Note: $A \times B \neq B \times A$. Why?

B×A contains things like (2, Aly) which is not the same as (Aly, 2)

Example: What is the Cartesian product $A \times B \times C$ where $A = \{a, b\}$, $B = \{x, y\}$, and $C = \{m, n\}$

$A \times B = \{(a, x, m), (a, x, n), (a, y, m), (a, y, n), (b, x, m), (b, x, n), (b, y, m), (b, y, n)\}$

Sets and Set Operations



Two or more sets can be combined in different ways.

Consider the sets of all CS majors and Applied Math majors at CU (sets C and A, respectively)

Def: Let A and B be sets. The union of the sets A and B, denoted $A \cup B$, is the set that contains those elements that are either in A or in B, or in both

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

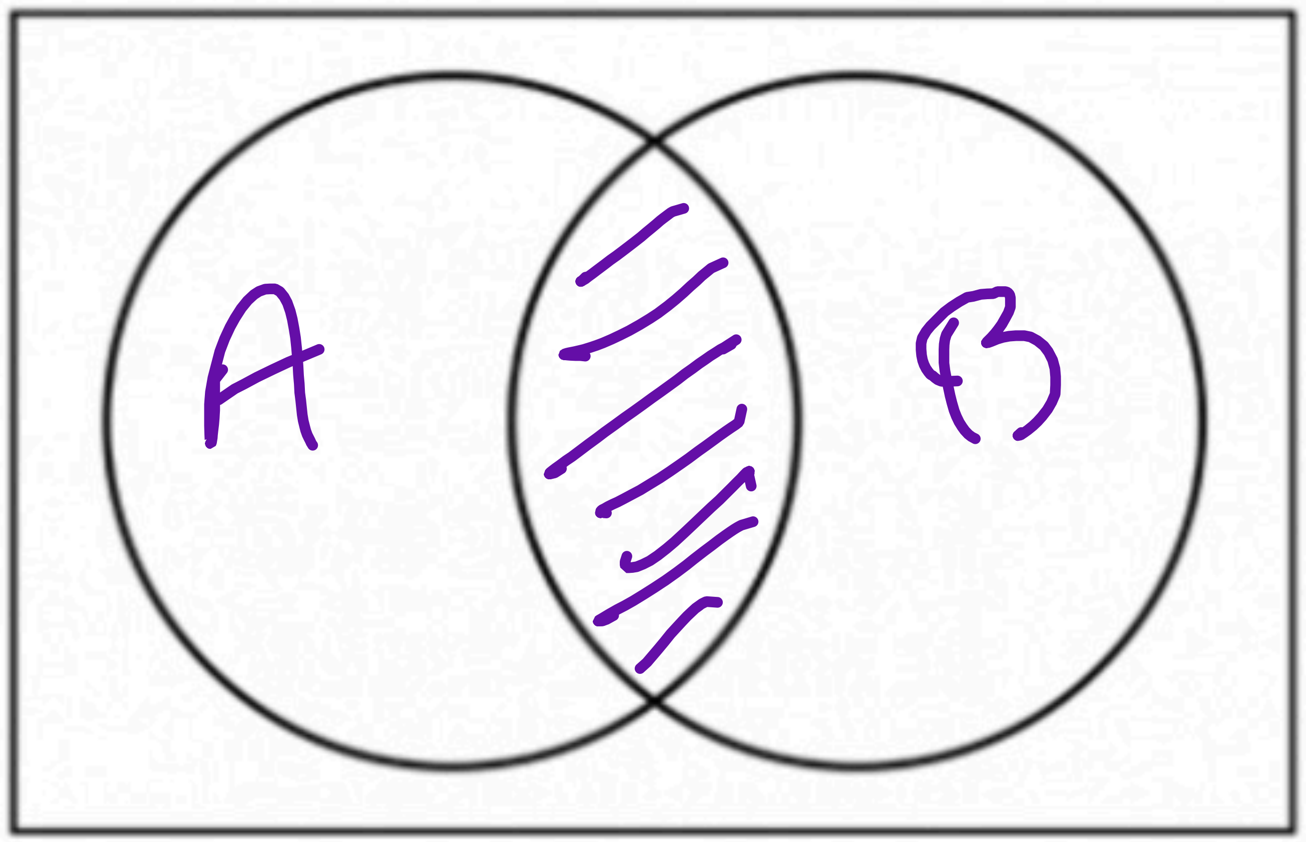
Example: The union of CS majors and Applied Math majors, $C \cup A$ is the set of all students that are either majoring in CS or in Applied Math

Sets and Set Operations

Def: Let A and B be sets. The **intersection** of the sets A and B, denoted $A \cap B$, is the set containing those elements in both A and B

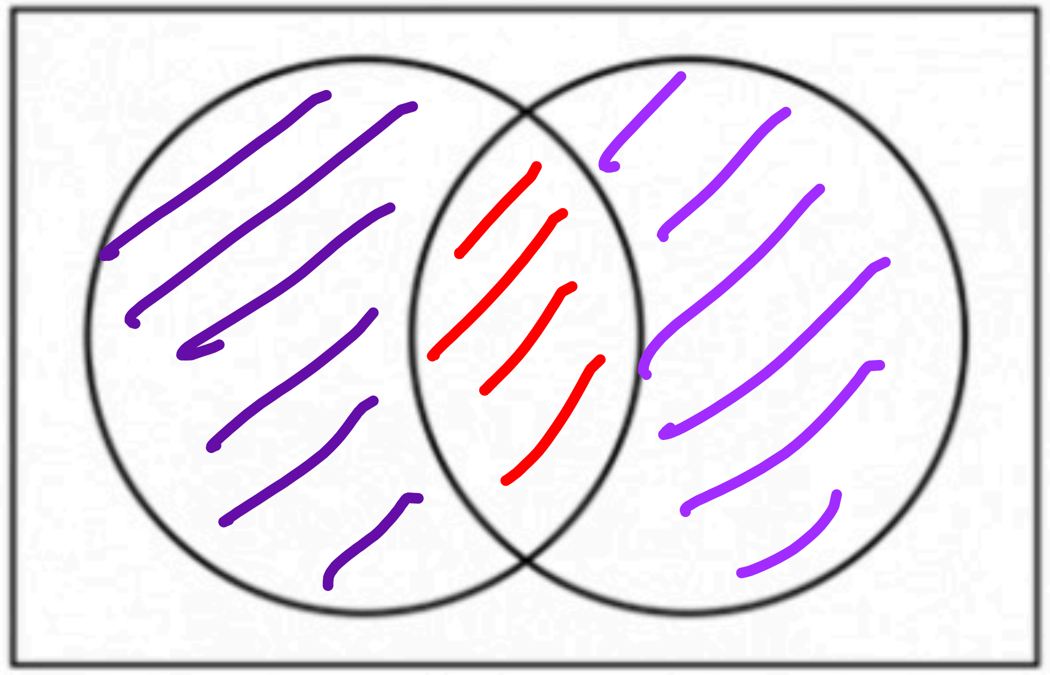
$$A \cap B = \{x \mid x \in A \land x \in B\}$$

Example: Describe the set $A \cap C$ from the previous example



Sets and Set Operations

Question: How many elements are there in $A \cup B$?



$|A \cup B| = |A| + |B| - |A \cap B|$

Things in intersection get counted twice so we have to remove them after!

Sets and Set Operations

Def: Let A and B be sets. The difference of A and B, denoted $A - B$ or $A \backslash B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A*

Question: How could you represent $A - B$ is set builder logic?

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

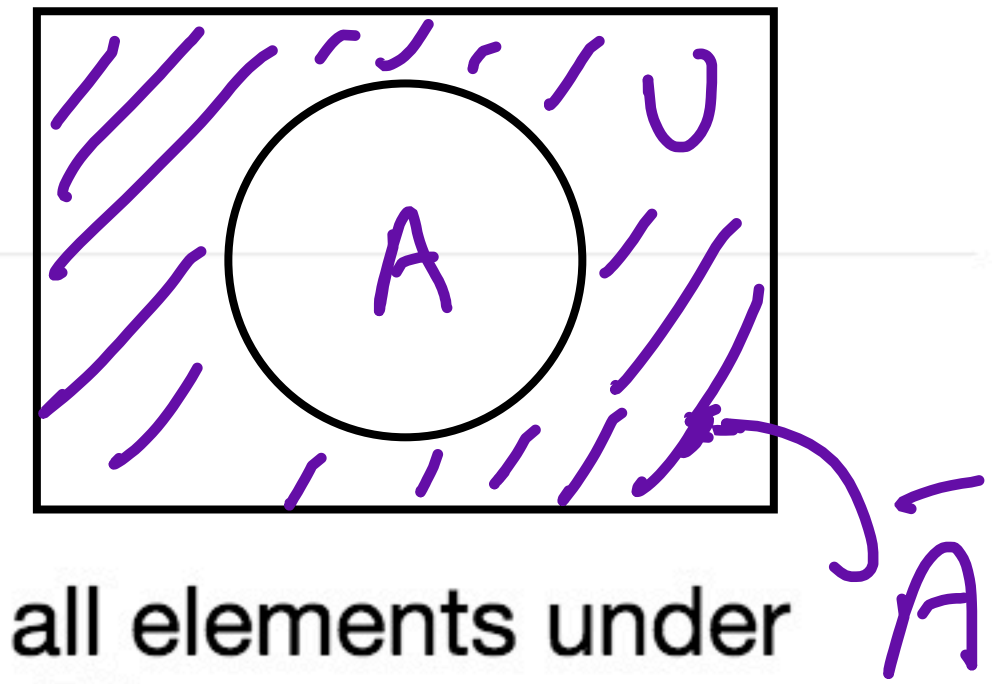
Question: What is the difference of the set of positive integers less than 10 and the set of prime numbers?

$$A = \{1, \not{2}, \not{3}, 4, \not{5}, 6, \not{7}, 8, 9\}$$

$$P = \{2, 3, 5, 7, 11, 13 \ldots\}$$

$$\Rightarrow A - P = \{1, 4, 6, 8, 9\}$$

Sets and Set Operations



Last One!

Lots of times we want to talk about the set of all elements under consideration, which we call the universal set, U.

Think of the universal set U as your domain of discourse

Def: Let U be the universal set. The **complement** of the set A, denoted $\bar{A}$, is the set $U - A$

An element belongs to $\bar{A}$ if and only if $x \notin A$, so

$$\bar{A} = \{x \in U \mid x \notin A\} \text{ or just } \{x \mid x \notin A\}$$