University of Colorado **Boulder**

Department of Computer Science
CSCI 2824: Discrete Structures

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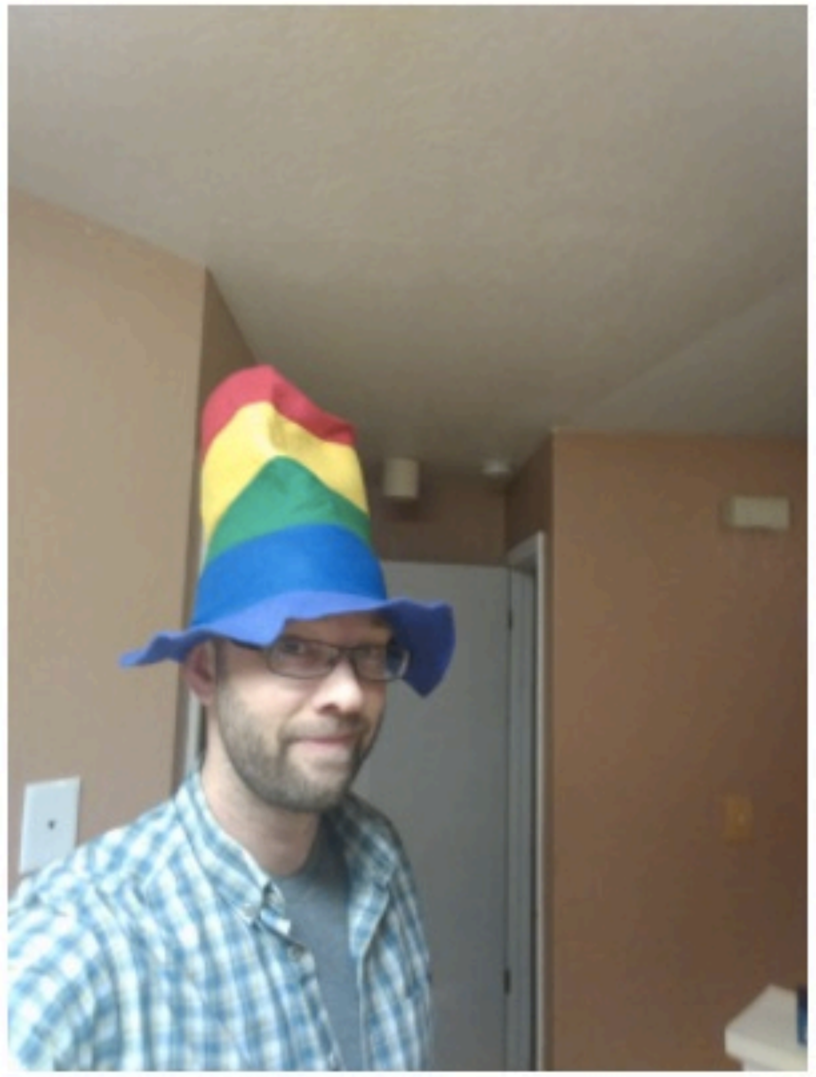
Lectures 11:
More on Set Operations and Cardinality of Infinite Sets

Announcements

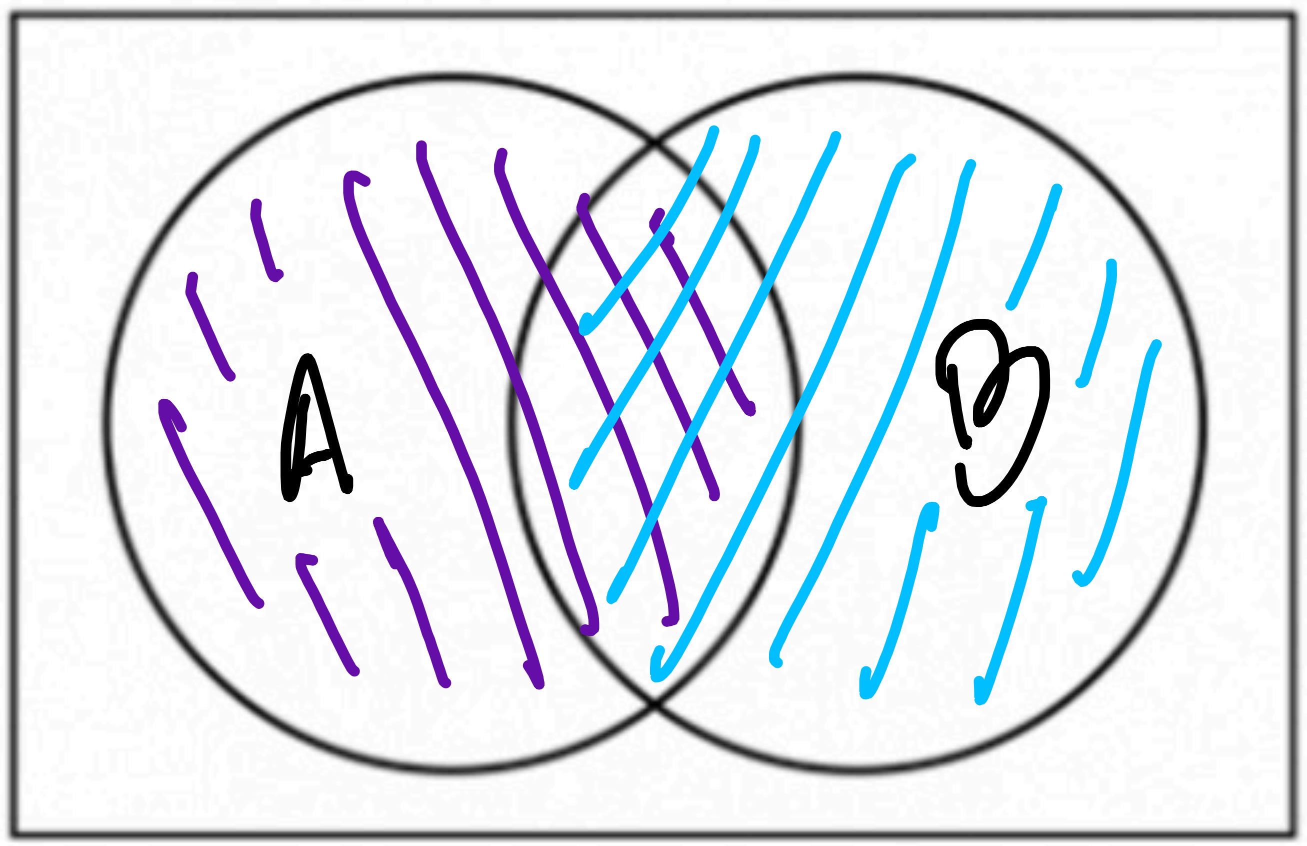
- Hmwk 4 posted. Due at the start of class on Friday 2/17
- By this afternoon, all CAs/Graders will have their hats, so look for them in CSEL



Sets and Set Operations

Def: Let A and B be sets. The **union** of the sets A and B, denoted $A \cup B$, is the set that contains those elements that are either in A or in B, or in both

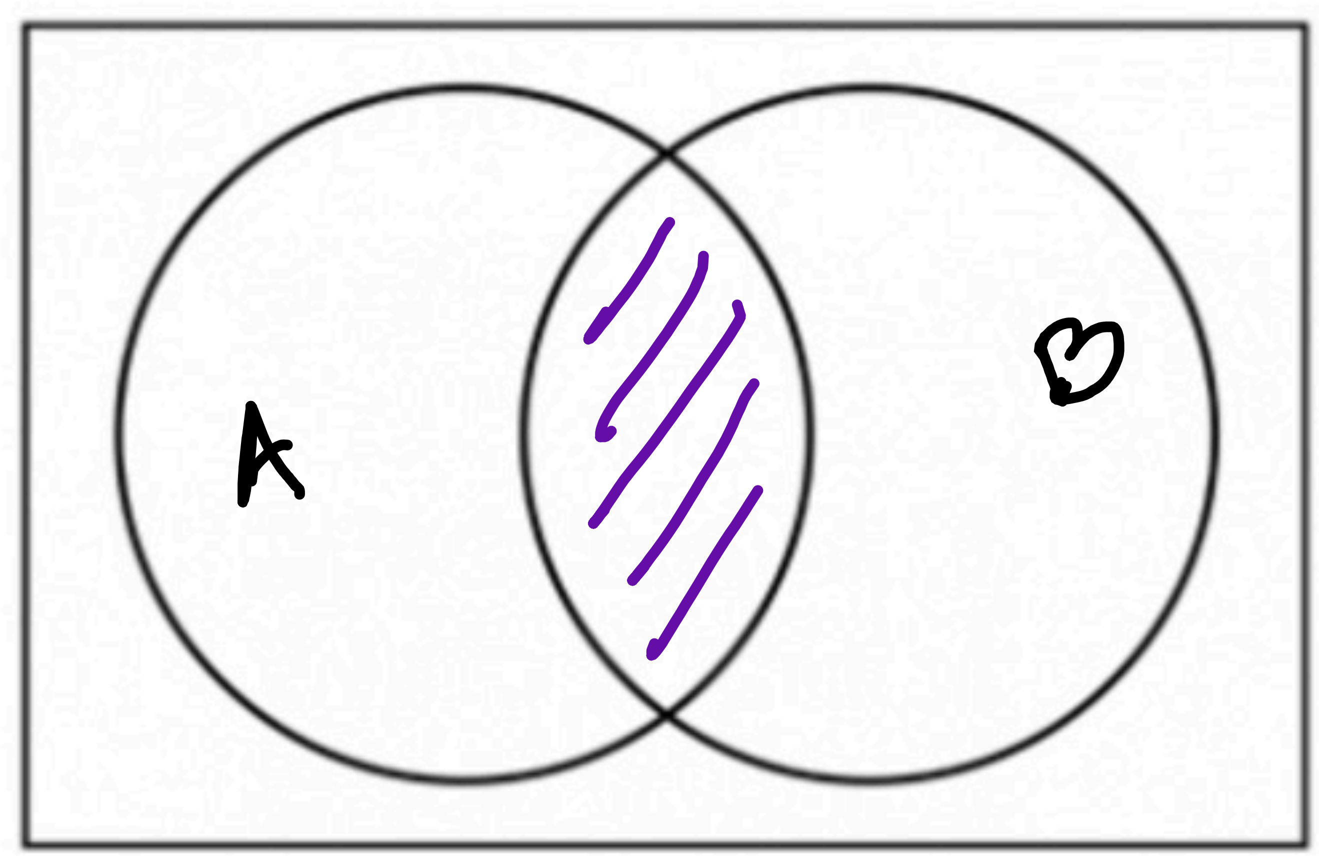
$$A \cup B = \{x \mid x \in A \vee x \in B\}$$



Sets and Set Operations

Def: Let A and B be sets. The **intersection** of the sets A and B, denoted $A \cap B$, is the set containing those elements in both A and B

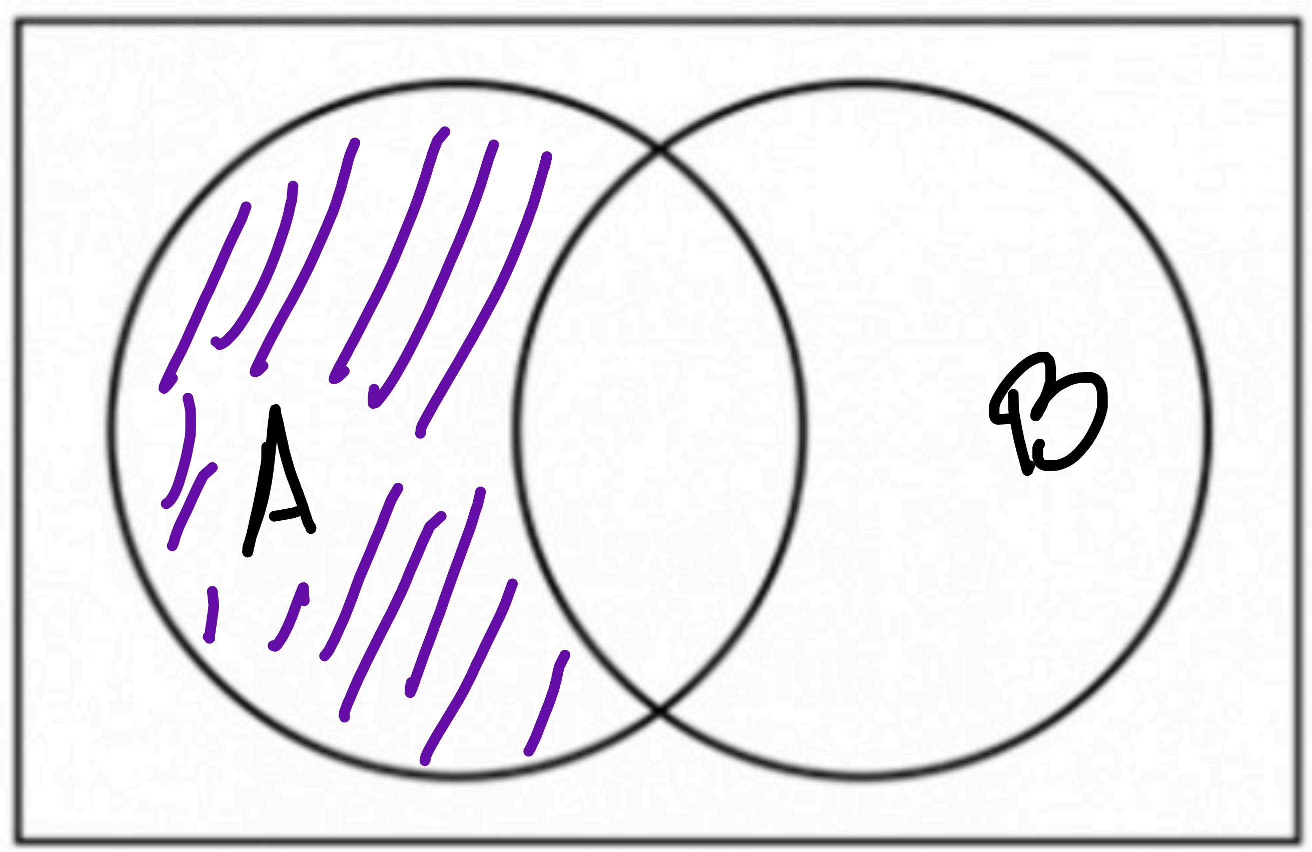
$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$



Sets and Set Operations

Def: Let A and B be sets. The **difference** of A and B, written as $A - B$, is the set containing those elements that are in A but not in B

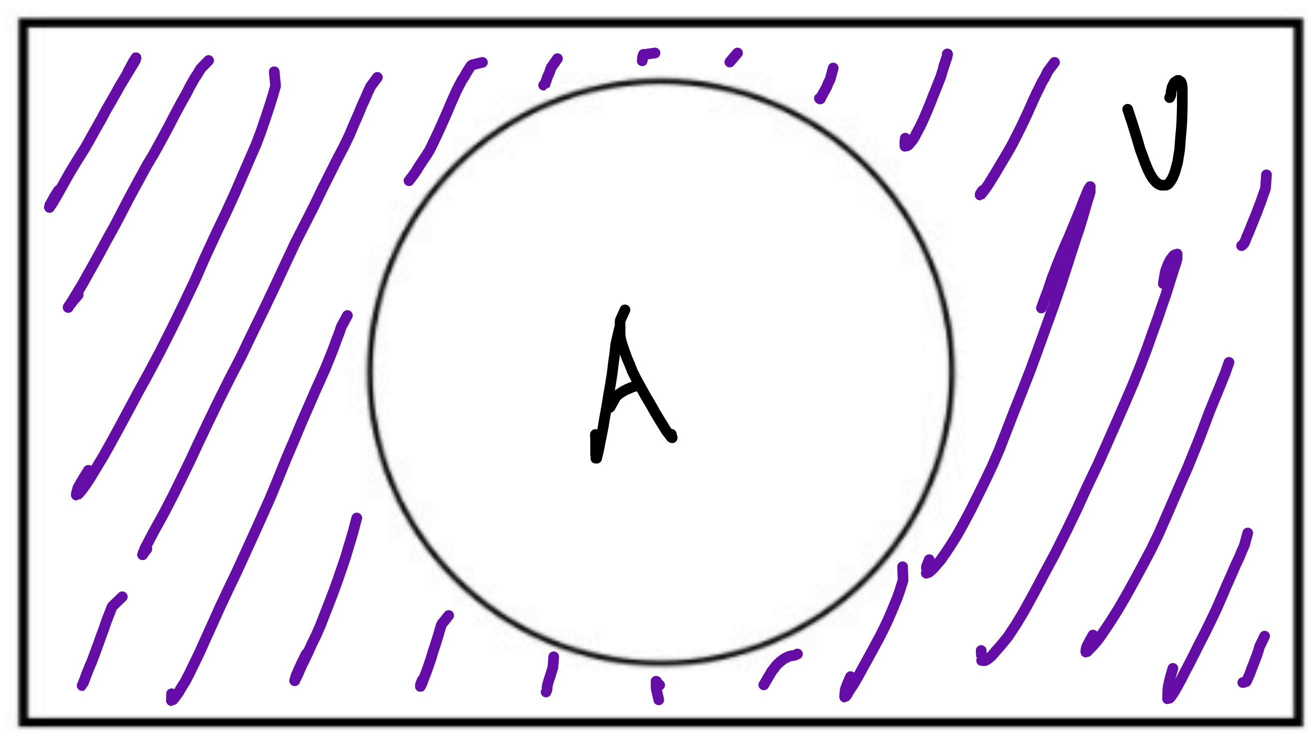
$$A - B = \{x \mid x \in A \wedge x \notin B\}$$



Sets and Set Operations

Def: Let U be the universal set. The **complement** of the set A, denoted $\bar{A}$, is the set $U - A$

$$\bar{A} = \{x \in U \mid x \notin A\} \text{ or just } \{x \mid x \notin A\}$$

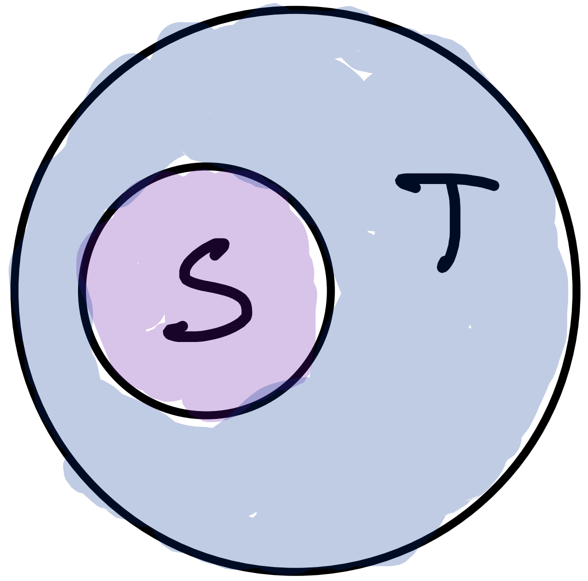


Sets and Set Operations

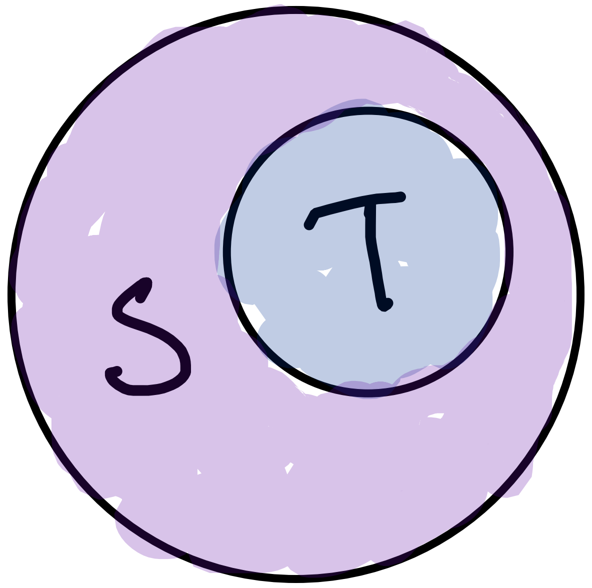
Sometimes, you want to prove that two complicated sets are equal

Strategy: To show that $S = T$, show that $S \subseteq T$ and $T \subseteq S$

1.($\Rightarrow$) To show $S \subseteq T$, assume $x \in S$ implies $x \in T$



2.($\Leftarrow$) To show $T \subseteq S$, assume $x \in T$ implies $x \in S$



Exercise: Show that $A - B = A \cap \bar{B}$

Sets and Set Operations

Exercise: Show that $A - B = A \cap \bar{B}$

Proof:

$1.$ (⇒) Let x be an arbitrary element in $A - B$.

This means that $x \in A$ and $x \notin B$

But $x \notin B$ implies that x is in the complement of B, i.e. $x \in \bar{B}$

Since $x \in A$ and $x \in \bar{B}$ we know $x \in A \cap \bar{B}$

Since x was any element in $A - B$ we've shown $A - B \subseteq A \cap \bar{B}$

Sets and Set Operations

Now we need to show that $A \cap \bar{B} \subseteq A - B$

2. (⇐) Let x be an arbitrary element in $A \cap \bar{B}$

This means that $x \in A$ and $x \in \bar{B}$

But $x \in \bar{B}$ implies that x is not in B, i.e. $x \notin B$

Since $x \in A$ and $x \notin B$ we know $x \in A - B$

Since x was any element in $A \cap \bar{B}$ we've shown $A \cap \bar{B} \subseteq A - B$

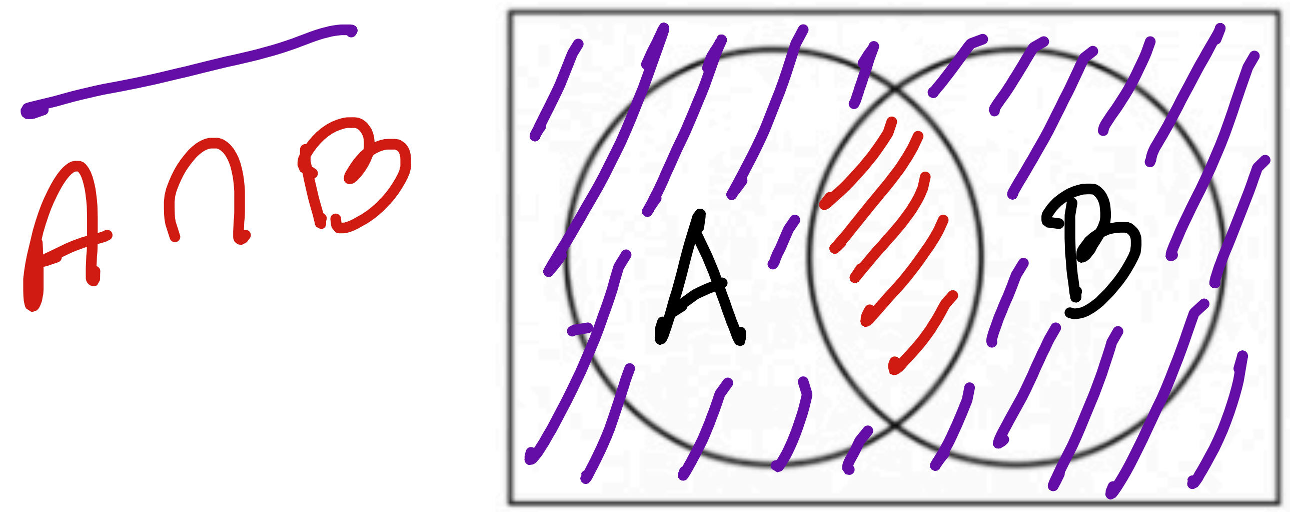
Since we've shown both $A - B \subseteq A \cap \bar{B}$ and $A \cap \bar{B} \subseteq A - B$

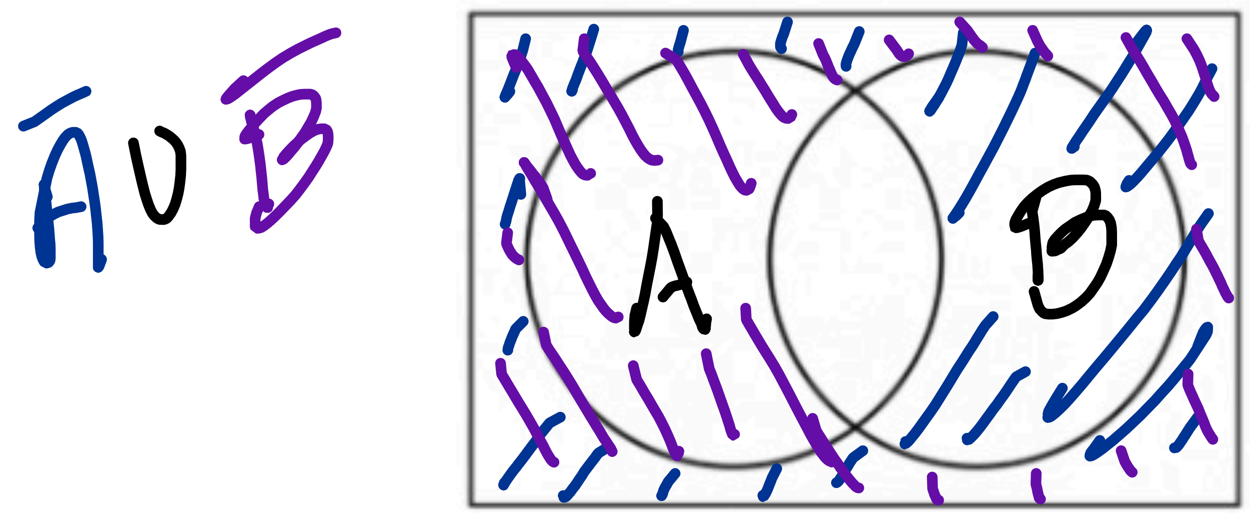
we've proved that $A - B = A \cap \bar{B}$

Sets and Set Operations

It turns out that when sets are combined using only ∪, ∩, and complements, there is this amazing symmetry between sets and set operations and propositional logic

Example: DeMorgan's Laws: $\overline{A \cap B} = \overline{A} \cup \overline{B}$





Sets and Set Operations

It turns out that when sets are combined using only ∪, ∩, and complements, there is this amazing symmetry between sets and set operations and propositional logic

Example: DeMorgan's Laws: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

To formally prove an identity such as this we could use the strategy

$$S \subseteq T \quad \text{and} \quad T \subseteq S \quad \Rightarrow \quad S = T$$

But often it's easier to use a proof based on set-builder notation

Sets and Set Operations

Example: DeMorgan's Laws: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Set Builder Proof: Using only logical equivalences

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{(def. complement)} \\
&= \{x \mid \neg(x \in A \cap B)\} && \text{(def. not in)} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{(def. intersection)} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{(DeMorgan's)} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{(def. not in)} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{(def. complement)} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{(def. union)} \\
&= \overline{A} \cup \overline{B}
\end{aligned}
$$

Sets and Set Operations

We of course have the other DeMorgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

EFY: Prove this identity using set builder notation

There are crap-ton more Set Identities that mirror logical equivalences. They're summarized on the table on the next slide.

From our logical definitions of Union, Intersection, and Complement we know that the natural logical equivalences are as follows:

$$\cup \quad \Leftrightarrow \quad \vee$$

$$\cap \quad \Leftrightarrow \quad \wedge$$

$$\text{complement} \quad \Leftrightarrow \quad \text{negation}$$

Sets and Set Operations

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Sets and Set Operations

Example: Use Set Identities to prove $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

$$
\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap \overline{(B \cap C)} && \text{(DeMorgan)} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{(DeMorgan)} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{(Commutativity)} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{(Commutativity)}
\end{aligned}
$$

EFY: Use Set Identities to prove $(A \cup \overline{B}) \cap (\overline{B \cap A}) = \overline{B}$

Sets and Set Operations

The equivalent set-version of a truth table is called a **membership table.** Can use to prove set equivalences

Example: Show that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

A	B	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$	$A \cup B$	$\overline{A \cup B}$
1	1	0	0	**0**	1	**0**
1	0	0	1	**0**	1	**0**
0	1	1	0	**0**	1	**0**
0	0	1	1	**1**	0	**1**

Identical columns in the membership table means that the two set expressions are equivalent

Sets and Set Operations

The equivalent set-version of a truth table is called a **membership table.** Can use to prove set equivalences

Example: Show that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

A	B	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$	$A \cup B$	$\overline{A \cup B}$
1	1	0	0	**0**	1	**0**
1	0	0	1	**0**	1	**0**
0	1	1	0	**0**	1	**0**
0	0	1	1	**1**	0	**1**

EFY: Use a membership table to show $(A \cup \overline{B}) \cap (\overline{B \cap A}) = \overline{B}$

Countable and Uncountable Sets

So far we've discussed sets like $A = \{a, b, c\}$ where, e.g. $|A| = 3$

Such a set is said to be a finite set or have finite cardinality

But we've not talked about the cardinality of sets like $\{n \in \mathbb{N} \mid n^2\}$ which clearly has an infinite number of elements

OK, so shouldn't the cardinality of $B = \{n \in \mathbb{N} \mid n^2\}$ be $|B| = \infty$ and we're done?

Countable and Uncountable Sets

OK, so shouldn't the cardinality of $B = \{n \in \mathbb{N} \mid n^2\}$ be $|B| = \infty$ and we're done?

Note quite, it turns out that it's useful to break up just **how infinite** a set is into two classes. Roughly they are described as follows:

Countably Infinite: We could count each member of the set if we had infinite time

Uncountable: We could never even list each element of the set even in infinite time

Countable and Uncountable Sets

OK, so shouldn't the cardinality of $B = \{n \in \mathbb{N} \mid n^2\}$ be $|B| = \infty$ and we're done?

Note quite, it turns out that it's useful to break up just **how infinite** a set is into two classes. Roughly they are described as follows:

Countably Infinite: We could count each member of the set if we had infinite time

Uncountable: We could never even list each element of the set even in infinite time

Def: A set A is called **countable** or **countably infinite** if it is not finite and there is a one-to-one map between each element of A and the natural numbers. A set A is called **uncountable** if it is infinite but not countable.

Countable and Uncountable Sets

Example: Show that the set of positive even integers is countable

We want to find a one-to-one map between the positive even integers $\{2, 4, 5, 8, 10, \ldots\}$ and the natural numbers $\{1, 2, 3, \ldots\}$

Countable and Uncountable Sets

Example: Show that the set of positive even integers is countable

We want to find a one-to-one map between the positive even integers $\{2, 4, 5, 8, 10, \ldots\}$ and the natural numbers $\{1, 2, 3, \ldots\}$

This one's pretty straightforward. We have

$\mathbb{N}$		Evens
1	$\Leftrightarrow$	2
2	$\Leftrightarrow$	4
3	$\Leftrightarrow$	6
	$\vdots$	

Better Yet: Define function relationship $f(n) = 2n$

Countable and Uncountable Sets

Example: Show that the set of all integers is countable

Countable and Uncountable Sets

Example: Show that the set of all integers is countable

We want to find a one-to-one map between the positive even integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ and the natural numbers

$\mathbb{N}$		~~Evens~~ Integers
1	$\Leftrightarrow$	0
2	$\Leftrightarrow$	1
3	$\Leftrightarrow$	−1
4	$\Leftrightarrow$	2
5	$\Leftrightarrow$	−2
	$\vdots$	

Better Yet: Define function relationship $f(n) = (-1)^n \lfloor n / 2 \rfloor$

Countable and Uncountable Sets

Example: The positive rational numbers are countable

$$
\begin{array}{ccccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \cdots \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \frac{6}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Countable and Uncountable Sets

Example: The real numbers are uncountable. Let's look at just $[0, 1]$

0.	1	2	3	4	3	5	3	4	5	3	0	8	⋯
0.	9	8	0	8	0	8	0	9	0	9	0	9	⋯
0.	7	5	0	0	3	8	4	2	3	4	0	8	⋯
0.	0	8	2	3	4	0	8	2	4	3	0	8	⋯
0.	5	9	8	2	3	6	1	5	3	8	9	4	⋯
0.	8	9	2	4	7	8	2	3	4	6	5	9	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Let's suppose we can list them all, and **look for a contradiction**

Countable and Uncountable Sets

Example: The real numbers are uncountable. Let's look at just $[0, 1]$

$$
\begin{array}{cccccccccccccc}
0. & 1 & 2 & 3 & 4 & 3 & 5 & 3 & 4 & 5 & 3 & 0 & 8 & \cdots \\
0. & 9 & 8 & 0 & 8 & 0 & 8 & 0 & 9 & 0 & 9 & 0 & 9 & \cdots \\
0. & 7 & 5 & 0 & 0 & 3 & 8 & 4 & 2 & 3 & 4 & 0 & 8 & \cdots \\
0. & 0 & 8 & 2 & 3 & 4 & 0 & 8 & 2 & 4 & 3 & 0 & 8 & \cdots \\
0. & 5 & 9 & 8 & 2 & 3 & 6 & 1 & 5 & 3 & 8 & 9 & 4 & \cdots \\
0. & 8 & 9 & 2 & 4 & 7 & 8 & 2 & 3 & 4 & 6 & 5 & 9 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Let's suppose we can list them all, and **look for a contradiction**

Contradiction: We'll **construct** a number that can't be in the list

Countable and Uncountable Sets

Example: The real numbers are uncountable. Let's look at just $[0, 1]$

0.	1	2	3	4	3	5	3	4	5	3	0	8	⋯
0.	9	8	0	8	0	8	0	9	0	9	0	9	⋯
0.	7	5	0	0	3	8	4	2	3	4	0	8	⋯
0.	0	8	2	3	4	0	8	2	4	3	0	8	⋯
0.	5	9	8	2	3	6	1	5	3	8	9	4	⋯
0.	8	9	2	4	7	8	2	3	4	6	5	9	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Let's suppose we can list them all, and **look for a contradiction**

Contradiction: We'll **construct** a number that can't be in the list

Strategy: Set the k^{th} digit of our new number based on the k^{th} digit of the k^{th} number in list according to a rule

Countable and Uncountable Sets

Example: The real numbers are uncountable. Let's look at just $[0, 1]$

0.	1	2	3	4	3	5	3	4	5	3	0	8	⋯
0.	9	8	0	8	0	8	0	9	0	9	0	9	⋯
0.	7	5	0	0	3	8	4	2	3	4	0	8	⋯
0.	0	8	2	3	4	0	8	2	4	3	0	8	⋯
0.	5	9	8	2	3	6	1	5	3	8	9	4	⋯
0.	8	9	2	4	7	8	2	3	4	6	5	9	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Rule:

- If k^{th} digit of the k^{th} number is a 3, our number's is a 5
- If k^{th} digit of the k^{th} number is not a 3, our number's is a 3

Countable and Uncountable Sets

Example: The real numbers are uncountable. Let's look at just $[0, 1]$

0.	1	2	3	4	3	5	3	4	5	3	0	8	⋯
0.	9	8	0	8	0	8	0	9	0	9	0	9	⋯
0.	7	5	0	0	3	8	4	2	3	4	0	8	⋯
0.	0	8	2	3	4	0	8	2	4	3	0	8	⋯
0.	5	9	8	2	3	6	1	5	3	8	9	4	⋯
0.	8	9	2	4	7	8	2	3	4	6	5	9	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Rule:

- If k^{th} digit of the k^{th} number is a 3, our number's is a 5
- If k^{th} digit of the k^{th} number is not a 3, our number's is a 3

$$m = 0\,.\,3\,3\,3\,5\,5\,3\,\cdots$$

Countable and Uncountable Sets

Claim: Our constructed number, m, can't already be in the list

Argument:

1. m isn't the 1^{st} number b/c their 1^{st} digits don't match
2. m isn't the 2^{nd} number b/c their 2^{nd} digits don't match
3. m isn't the 3^{rd} number b/c their 3^{rd} digits don't match

and so on and so on ..

Thus we've constructed an m that can't be in the list

This is our contradiction that proves that the real numbers in $[0, 1]$ are uncountable

This proof is called **Cantor's Diagonal Argument**

EFYs

Sets and Set Operations

We of course have the other DeMorgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

EFY: Prove this identity using set builder notation

$$
\begin{aligned}
\overline{A \cup B} &= \{x \mid x \notin A \cup B\} && \text{(def. complement)} \\
&= \{x \mid \neg(x \in A \cup B)\} && \text{(def. not in)} \\
&= \{x \mid \neg(x \in A \vee x \in B)\} && \text{(def. intersection)} \\
&= \{x \mid \neg(x \in A) \wedge \neg(x \in B)\} && \text{(DeMorgan's)} \\
&= \{x \mid x \notin A \wedge x \notin B\} && \text{(def. not in)} \\
&= \{x \mid x \in \overline{A} \wedge x \in \overline{B}\} && \text{(def. complement)} \\
&= \{x \mid x \in \overline{A} \cap \overline{B}\} && \text{(def. union)} \\
&= \overline{A} \cap \overline{B}
\end{aligned}
$$

Sets and Set Operations

EFY: Use Set Identities to prove $(A \cup \overline{B}) \cap (\overline{B \cap A}) = \overline{B}$

$$
\begin{aligned}
(A \cup \overline{B}) \cap (\overline{B \cap A}) &= (A \cup \overline{B}) \cap (\overline{B} \cup \overline{A}) && \text{(DeMorgan)} \\
&= (A \cup \overline{B}) \cap (\overline{A} \cup \overline{B}) && \text{(Comm.)} \\
&= (A \cap \overline{A}) \cup \overline{B} && \text{(Distribution)} \\
&= \emptyset \cup \overline{B} && \text{(Complement)} \\
&= \overline{B} && \text{(Identity)}
\end{aligned}
$$

Sets and Set Operations

EFY: Use a membership table to show $(A \cup \overline{B}) \cap (\overline{B \cap A}) = \overline{B}$

A	B	$\overline{B}$	$A \cup \overline{B}$	$B \cap A$	$\overline{B \cap A}$	$(A \cup \overline{B}) \cap (\overline{B \cap A})$
1	1	**0**	1	1	0	**0**
1	0	**1**	1	0	1	**1**
0	1	**0**	0	0	1	**0**
0	0	**1**	1	0	1	**1**

The columns of interest are identical, so the two sets are equal.