University of Colorado **Boulder**

Department of Computer Science

CSCI 2824: Discrete Structures

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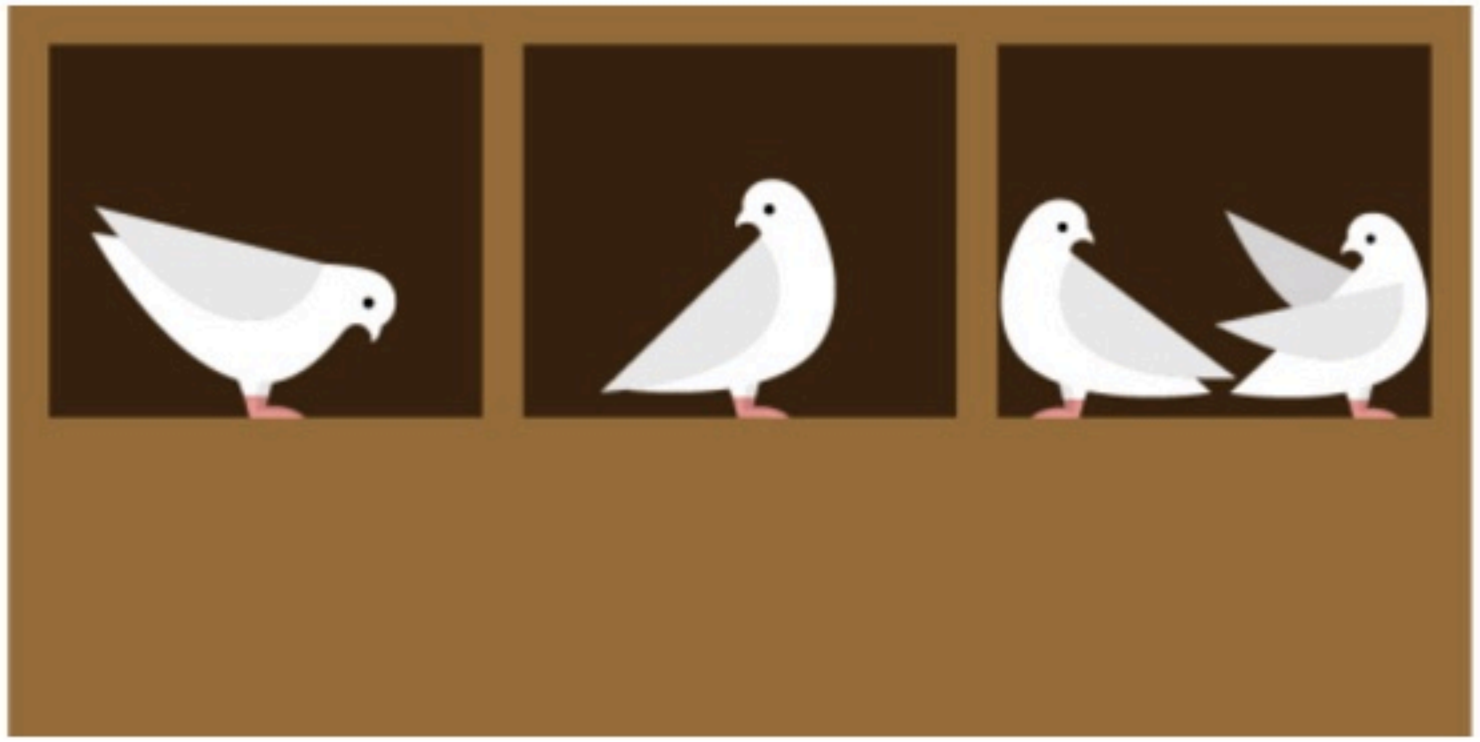
Lecture 27
The Pigeonhole Principle

Pigeonhole Principle

Suppose that a flock of 4 pigeons flies into a set of 3 pigeonholes to roost. Because there are 4 pigeons and only 3 pigeonholes for them to go into, at least one of the pigeonholes must have at least two pigeons in it.



The Pigeonhole Principle: If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Pigeonhole Principle

Example: A drawer contains a dozen brown socks and a dozen black socks, all unmatched. If you take out socks in the dark, how many must you grab to ensure that you have two socks that match?

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Since there are two colors of socks, the Pigeonhole Principle tells us that we must remove $2 + 1 = 3$ socks to get two of the same color

Example: How many socks must we remove to guarantee that we get 2 black socks?

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Example: How many socks must we remove to guarantee that we get 2 black socks?

We must remove 14 socks, because we could draw the first 12 brown socks and then when we draw 2 more we'd have 2 black socks.

Pigeonhole Principle

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There are 26 letters in the alphabet.

By the Pigeonhole Principle, since there are more than 27 last names under consideration, at least two of them must start with the same letter

Pigeonhole Principle

Example: How many cards must be drawn from a standard deck of 52 cards to ensure that at least 3 of the cards are of the same suit

Pigeonhole Principle

Example: How many cards must be drawn from a standard deck of 52 cards to ensure that at least 3 of the cards are of the same suit

Strategy: Think how many cards we could distribute in a particular way such that no 3 of the cards are of the same suit

We could have 2 hearts, 2 diamonds, 2 clubs, and 2 spades

As soon as we add 1 more card there will be three of the same suits

So the answer is 9 cards

Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Pigeonhole Principle

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Card Example: The 4 suits are the boxes.

If we have $N = 9$ cards then there is at least one suit with at least

$$\left\lceil \frac{9}{4} \right\rceil = \lceil 2.25 \rceil = 3 \text{ cards of the same suit}$$

Pigeonhole Principle

Example: Show that there are at least seven people in California (pop. 38.8 million) with the same three initials that were born on the same day of the year. (Assume that everyone has three initials)

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There are $26^3 \times 365 = 6,415,240$ initials-birthday combinations

By the Generalized Pigeonhole Principle, there are at least

$$\left\lceil \frac{38800000}{6415240} \right\rceil = \lceil 6.05 \rceil = 7$$

people with the same initials-birthday combination

Pigeonhole Principle

Example: Show that for any number n there is a multiple of n that has only 0's and 1's in it's decimal representation

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Consider the $n+1$ numbers $1, 11, \dots, 11\cdots1$ where the last number has $n+1$ ones in it

Note that if we divide a number by n there are only n possible remainders $(0, 1, \dots n-1)$

Since there are $n+1$ numbers above, at least 2 of them must have the same remainder when divided by n. Let those two numbers be a and b, where $b > a$

Then $n \mid (b-a)$ so $(b-a)$ is a multiple of n

Finally, $(b-a)$ is made up only of the digits 0 and 1

Pigeonhole Principle

Example: Show that for any number n there is a multiple of n that has only 0's and 1's in it's decimal representation

For instance, if we let $n = 17$

$$1 \textbf{ mod } 17 = 1 \quad \text{and} \quad 1111111111111111 \textbf{ mod } 17 = 1$$

So 11111111111111110 is divisible by 17

In fact, $11111111111111110 = 17 \times 653594771241830$

Pigeonhole Principle

EFY: Show that if a function f maps elements from a set of $k+1$ or more elements to a set with k elements then f is not one-to-one.

EFY: Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n

EFY: What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

EFY: There are 38 different time slots during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

EFYs