Numerical Errors

Need ways of measuring size when there is more than one variable (higher dimensional problem

So, we need an excursion

Let V be a vector space over $\mathbb{R}$ (might be $\mathbb{R}^n$, or space of matrices or a function space, etc)

A norm on V is a function $V \longrightarrow \mathbb{R} : v \longmapsto \|v\|$ with the following properties:

(1) $\forall v \in V, \ \|v\| \geq 0$ and $\|v\| = 0 \Leftrightarrow v = 0$

(2) if $\lambda \in \mathbb{R}, \ v \in V$ then $\|\lambda v\| = |\lambda| \, \|v\|$

(3) if $v_1, v_2 \in V$ then $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$

this is called the triangle inequality

I'd call it part 1

if $n=1$, $|\cdot|$ (abs value) on $\mathbb{R}$

Examples on $\mathbb{R}^n = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| x_i \in \mathbb{R} \right\}$

$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$

Box norm, or "infinity norm"

$\|x\|_\infty = \|x\|_b = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$

Euclidean norm, a.k.a. 2-norm

$\|x\|_2 = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$

It takes some work to show this is a norm.

2nd part of triangle inequality

$\|\cdot\|$ is a norm on V

then

$$\big| \|v_1\| - \|v_2\| \big| \leq \|v_1 \pm v_2\|$$

Pf
$$\|v_1\| = \|v_1 - v_2 + v_2\|$$
$$\leq \|v_1 - v_2\| + \|v_2\|$$

So $\|v_1\| - \|v_2\| \leq \|v_1 - v_2\|$

Switching the letters

$$\|v_2\| - \|v_1\| \leq \|v_2 - v_1\|$$

Note that

$$\| -v \| = \| (-1)v \| = |-1| \|v\| = \|v\|$$

So

$$\| u - v \| = \| -(v-u) \| = \| v - u \|$$

Thus,

$$\|v_1\| - \|v_2\| \leq \|v_1 - v_2\|$$

$$\|v_2\| - \|v_1\| \leq \|v_1 - v_2\|$$

So

$$\big| \|v_1\| - \|v_2\| \big| \leq \|v_1 - v_2\|$$

Replacing v_2 by $(-v_2)$

gives

$$\big| \|v_1\| - \underbrace{\|-v_2\|}_{=\|v_2\|} \big| \leq \|v_1 + v_2\|$$

So

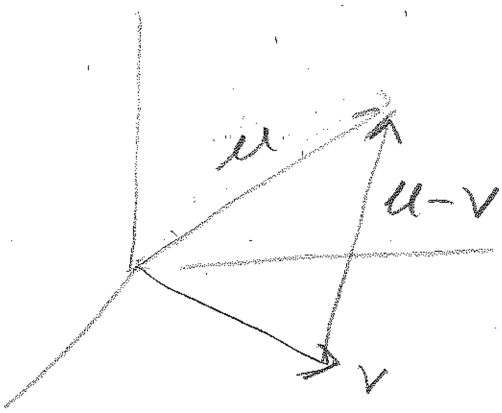
$$\big| \|v_1\| - \|v_2\| \big| \leq \|v_1 + v_2\|$$

Similarly

$$\| v_1 - v_2 \| \leq \|v_1\| + \|v_2\|$$

Intuitively, $\|v\|$ is the "size" of the vector v.
and
$d(u,v) = \|u - v\|$ is the distance between u and v.

picture in $\mathbb{R}^3$



<u>Equivalence of norms</u>

If $\|\cdot\|_1$ and $\|\cdot\|_2$ (not the examples in $\mathbb{R}^n$) are norms on V, we say they are <u>equivalent</u> if there are constants A and B such that

$$\forall v \in V \quad \|v\|_1 \leq A\|v\|_2$$

$$\text{and} \quad \|v\|_2 \leq B\|v\|_2 .$$

<u>Exercise</u> Show this is an equivalence relation on the set of norms on V

For many purposes a norm can be replaced by an equivalent norm without changing the statement in question.

<u>Examples on $\mathbb{R}^n$</u>

Consider $\|\cdot\|_2$, $\|\cdot\|_1$, $\|\cdot\|_\infty$

Note, if $x \in \mathbb{R}^n$

$$x_i^2 \leq x_1^2 + \cdots + x_{i-1}^2 + x_i^2 + x_{i+1}^2 + \cdots + x_n^2$$

so $x_i^2 \leq \|x\|_2^2$, taking $\sqrt{\ }$ gives

$$|x_i| = \sqrt{x_i^2} \leq \|x\|_2$$

<u>Claim</u> All of these norms are equivalent

Consider $\|\cdot\|_1$ and $\|\cdot\|_\infty$

$$|x_i| \leq \max\{|x_1|, \ldots, |x_n|\} = \|x\|_\infty$$

So

$$\underbrace{|x_1| + |x_2| + \cdots + |x_n|}_{= \|x\|_1} \leq \underbrace{\|x\|_\infty + \cdots + \|x\|_\infty}_{n \text{ terms}} = n\|x\|_\infty$$

So $\|x\|_1 \le n\|x\|_b$

Since $|x_i| \le |x_1| + \cdots + |x_n| = \|x\|_1$,

$\|x\|_b = \max\{|x_1|, \ldots, |x_n|\} \le \|x\|_1$

Thus
$$\begin{cases} \|x\|_1 \le n\|x\|_b \\ \|x\|_b \le \|x\|_1 \end{cases}$$
so they are ~~equivalent~~

Consider $\|\cdot\|_b$ and $\|\cdot\|_2$

Since $|x_i| \le \|x\|_2$,

$\|x\|_b \le \|x\|_2$

We have
$$\|x\|_2^2 = x_1^2 + \cdots + x_n^2 = |x_1|^2 + \cdots + |x_n|^2$$
$$\le \underbrace{\|x\|_b^2 + \cdots + \|x\|_b^2}_{n \text{ terms}}$$
$$= n\|x\|_b^2$$

So $\|x\|_2 \le \sqrt{n}\,\|x\|_b$

Thus $\begin{cases} \|x\|_6 \le \|x\|_2 \\ \|x\|_2 \le \sqrt{n}\, \|x\|_6 \end{cases}$

Exercise Show from what we have here that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Thm: all norms on a finite dimensional vector space are equivalent, the space is complete is any of those norms

Operator norms Let $L : U \to V$ be a linear transformation U and V are normed spaces

Defn: L is bounded if there is a constant C so that $\|L(u)\|_V \le C\|u\|_U$

for all $u \in U$

Thm if U and V are finite dimensional all linear transformations $U \to V$ are bounded pf: $L: U \to V$

Choose a basis $u_1, \dots, u_n$ of U
So, we have a linear isomorphism
$\varphi : \mathbb{R}^n \longrightarrow U$ by

$$\varphi(c) = c_1 u_1 + \cdots + c_n u_n$$

If $u = c_1 u_1 + \cdots + c u_n$, define

$$\|u\|_* = \max\{|c_1|, \dots, |c_n|\}$$
$$= \|\varphi^{-1}(u)\|_b$$

Exercise: $\|u\|_*$ is a norm on U

If $u \in U$, $u = c_1 u_1 + \cdots + c_n u_n$

$\|\cdot\|_V$ $$\|L(u)\| = \|c_1 L(u_1) + \cdots + c_n L(u_n)\|$$
$$\leq \|L(u_1)\||c_1| + \cdots + \|L(u_n)\||c_n|$$
$$\leq \underbrace{\{\|L(u_1)\| + \cdots + \|L(u_n)\|\}}_{\text{call it } C} \|u\|_*$$

so
$$\|L(u)\| \leq C\|u\|_*$$

Since all norms on V are equivalent there is a constant A such that

$$\|u\|_* \le A\|u\|_V$$

So $\|L(u)\|_V \le C'A\|u\|_V$

Suppose L is bounded

We want to find the smallest C' so that $\|L(u)\| \le C'\|u\| \quad \forall u \in V$

If C' is some constant so that $\|L(u)\| \le C'\|u\|$

and $u \neq 0$ $\quad \frac{1}{\|u\|}\|L(u)\| \le C'$

$$\Big\|\Big|\frac{1}{\|u\|}\Big|\,\|L(u)\|$$

$$\Big\|\frac{L(u)}{\|u\|}\Big\|$$

Define $\|L\|$ (notation gives the problem away) 60

$$\|L\| = \sup\left\{ \frac{\|L(u)\|}{\|u\|} \,\middle|\, u \in V,\ u \neq 0 \right\}$$

Then $\|L\| \leq C$

Exercise:

$$\left\{ \|L(w)\| \,\middle|\, w \in V,\ \|w\| = 1 \right\}$$

is the same set of numbers as

$$\left\{ \frac{\|L(u)\|}{\|u\|} \,\middle|\, u \in V,\ u \neq 0 \right\}$$

So $\|L\| = \sup\left\{ \|L(w)\| \,\middle|\, w \in V,\ \|w\| = 1 \right\}$

Claim: $\|L(u)\| \leq \|L\| \, \|u\|$

prf if $u \neq 0$, then

$$\frac{\|L(u)\|}{\|u\|} \leq \sup\left\{ \frac{\|L(w)\|}{\|w\|} \,\middle|\, w \in V,\ w \neq 0 \right\}$$

$$= \|L\|$$

So $\dfrac{\|L(u)\|}{\|u\|} \leq \|L\|$

which gives us

$$\|L(u)\| \le \|L\| \, \|u\|$$

if $u \neq 0$. But it's surely also true if $u = 0$.

So $\|L(u)\| \le \|L\| \, \|u\| \quad \forall u$

and if $\|L(u)\| \le C\|u\| \quad \forall u$ then

$$\|L\| \le C$$

So $\|L\|$ is the smallest such constant.

<u>Claim</u> Let $L(U, V)$ be the space of linear transformations from U to V.

This is a vector space under the operations

$L_1 + L_2$ is defined by

$$(L_1 + L_2)(u) = L_1(u) + L_2(u)$$

and λL is defined by

$$(\lambda L)(u) = \lambda(L(u))$$

Then $\|L\|$ is a norm on $L(U; V)$

pf: (1) $\|L\| \geq 0$ by defn,

if Θ is the zero transformation

$(\Theta(u) = 0$ for all $u \in U)$

then $\|\Theta(u)\| = 0 \leq 0\|u\| \quad \forall u$

So $\|L\| \leq 0$.

Conversely, if $\|L\| = 0$

then $\|L(u)\| \leq 0\|u\| = 0 \quad \forall u$

So $\|L(u)\| = 0 \ \forall u$

$\Rightarrow L(u) = 0 \ \forall u$

$\Rightarrow L = \Theta$

(2) Suppose $\lambda \in \mathbb{R}$, $\lambda \neq 0$; $u \in U$

$$\|L(u)\| = \|\tfrac{\lambda}{\lambda} L(u)\|$$

$$= \|(\lambda L)(\tfrac{u}{\lambda})\| \leq \|\lambda L\| \, \|\tfrac{u}{\lambda}\|$$

$$= \|\lambda L\| \frac{1}{|\lambda|} \|u\|$$

$$= \frac{\|\lambda L\|}{|\lambda|} \|u\|$$

So $\|L\| \leq \frac{\|\lambda L\|}{|\lambda|}$

and $|\lambda| \, \|L\| \leq \|\lambda L\|$

This is certainly also true if $\lambda = 0$

Combining the inequalities gives

$$|\lambda| \, \|L\| = \|\lambda L\|$$

(3) (Triangle inequality)

$$\|(L_1 + L_2)(u)\| = \|L_1(u) + L_2(u)\|$$
$$\leq \|L_1(u)\| + \|L_2(u)\|$$
$$\leq \|L_1\| \, \|u\| + \|L_2\| \, \|u\|$$
$$= \left[\|L_1\| + \|L_2\| \right] \|u\|$$

So $\|L_1 + L_2\| \leq \|L_1\| + \|L_2\|$

Thm If we have bounded linear transformations

$$U \xrightarrow{S} V \xrightarrow{T} W$$

where U, V, W are normed spaces, then

$$(*) \quad \|TS\| \le \|T\|\,\|S\|$$

Pf
$$\|(TS)(u)\| = \|T(S(u))\|$$
$$\le \|T\|\,\|S(u)\|$$
$$\le \|T\|\,\|S\|\,\|u\|$$

So $\|TS\| \le \|T\|\,\|S\|$

Rmk $(*)$ is not necessarily true for arbitrary norms on the spaces $L(U;V)$ and $L(V;W)$.

If $\mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$, then $L(x) = Ax$ for a unique $m \times n$ matrix

$A = [L]_{std}$ is the standard matrix of L.

If $\mathbb{R}^{m\times n}$ is the space of $m\times n$ matrices with real entries

($\mathbb{R}^n = \mathbb{R}^{n\times 1}$)

We have $L(\mathbb{R}^n;\mathbb{R}^m) \longrightarrow \mathbb{R}^{m\times n}$

$L \longmapsto [L]_{std}$

This is a linear iso.

We cheerfully conflate the linear transformation and its matrix.

So, if we choose some norms $\|\cdot\|_{\mathbb{R}^n}$, $\|\cdot\|_{\mathbb{R}^m}$ on $\mathbb{R}^n$ and $\mathbb{R}^m$

we have a norm $\|\cdot\|$ on $\mathbb{R}^{m\times n}$

by $\|[L]_{std}\| = \|L\|$.

You get different norms on $\mathbb{R}^{n\times m}$ if you choose different norms on $\mathbb{R}^n$ and $\mathbb{R}^m$

Box norms on $\mathbb{R}^n, \mathbb{R}^m$

$$\cancel{y =} \; Ax$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$y_j = \sum_{k=1}^{n} a_{jk} x_k$$

So
$$|y_j| = \left| \sum_k a_{jk} x_k \right|$$
$$\leq \sum_k |a_{jk}| |x_k|$$
$$\leq \sum_k |a_{jk}| \, \|x\|_b$$
$$= \underbrace{\left[\sum_k |a_{jk}| \right]}_{\text{Row sum on } j^{th} \text{ row.}} \|x\|_b$$
$$\leq \|A\|_{bb} \|x\|_b$$

where $\|A\|_{bb} = \max\limits_{j=1,\dots,m} \left(\sum_{k=1}^{n} |a_{jk}| \right)$

Ex. Check this is a norm.

$\|A\|_{op} \leq \|A\|_{bb}$

<u>Claim</u> $\|A\|_{bb}$ is the operator norm $\|A\|_{op}$ of the linear transformation

<u>pf</u> Select i so that the i^{th} row sum is the max, say $M = \sum_k |a_{ik}| = \|A\|_{bb}$.

For each element a_{ik} we can pick $\omega_k = \pm 1$ so that $\omega_k a_{ik} = |a_{ik}|$

Let $\omega = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix} \in \mathbb{R}^n$

So $\|\omega\|_b = \max\{ |\omega_k| \mid k = 1, \ldots, n\}$
$= 1$

Then if $y = A\omega$
$y_i = \sum a_{ik}\omega_k = \sum |a_{ik}| = M$

So $|y_i| = M$ (y_i is positive)

So $\|A\omega\|_b = \max\{|y_1|, \ldots, |y_m|\}$
$\geq |y_i| = M$.

Thus $M \le \|Aw\|_b \le \|A\|_{op} \|w\|_b = \|A\|_{op}$

So $\|A\|_{bb} \le \|A\|_{op}$ QED

Problem If we put the one norms on $\mathbb{R}^n, \mathbb{R}^m$, what is $\|A\|_{op}$?

The following prop ~~takes more~~ work:

Thm Put the Euclidean norms $\|\cdot\|_2$ on $\mathbb{R}^n$ and $\mathbb{R}^m$ and let A be a matrix, considered as a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$

The $\|A\|_{op} = \sqrt{\lambda^*}$,

where λ^* is the largest eigenvalue of A^TA.

Rmk: A^TA is $n \times n$ and its eigenvalues are real and non-negative.

Rmk If $m=n$, and $A: \mathbb{R}^n \to \mathbb{R}^n$ $\|A\|_{op}$ is $\max\{|\lambda_i| \mid \lambda_i \in \mathrm{Spec}(A)\}$

(the eigenvalues could be complex)

<u>Quick Review</u>: Let f, g be functions on $\mathbb{R}$. Consider what happens as $x \to L$ $(L \in [-\infty, \infty])$ then

$$f(x) = O(g(x))$$

means there is a constant $C \geq 0$ so that

$$|f(x)| \leq C\,|g(x)|$$

on some nbhd of L. ~~alternate~~ Equivalently (if $g(x) \neq 0$ on the nbhd)

$$\left|\frac{f(x)}{g(x)}\right| = \frac{|f(x)|}{|g(x)|} \text{ is bounded}$$

people often write "$O(g(x))$" to mean some function with the property. So, for example

if $f_1(x) = O(g(x))$

$f_2(x) = O(g(x))$

Then $f_1(x) + f_2(x) = O(g(x)) + O(g(x))$

$= O(g(x))$

Similarity $f(x) = o(g(x))$

means $\frac{f(x)}{g(x)} \to 0$ as $x \to L$

or (better)

$f(x) = g(x)\varphi(x)$ where $\varphi(x)$ is some function such that

$\varphi(x) \to 0$ as $x \to L$