(Review of) Linear Algebra

UP201

IISc Bangalore

Ok, the Questions

- Why do they say “abstract” algebra?
- Why bother about all this “abstract” stuff?
- What is a vector, any way?
- And what is a matrix?
- Eigenvalues, eigenvectors??

Plan of Review

- What is abstract about "abstract algebra"?
- Sets
- Vectors Spaces
- Basis
- Representations
- Operators
- Eigenvectors and Eigenvalues

What is abstract about "abstract algebra"?

- Abstract Algebra = *Abstracted* Algebra
- Abstracted from Physics, if you like!
- It *is not abstract*!

Sets

- A very basic idea in modern mathematics
- Set – “Collection of well defined objects”
- Give examples
- It is possible to define binary operations such as “addition” on sets
- Sometimes “space” is used to mean a set

Vector Space

A set $\mathcal{V} = \{\boldsymbol{v} | \boldsymbol{v} \text{ is a vector}\}$ is called a vector space if there is a binary operation '+' (associative) and scalar multiplication which satisfy

- $\forall \boldsymbol{v} \in \mathcal{V}$, $\alpha\boldsymbol{v} \in \mathcal{V}$ is also a vector $\forall$ numbers α
- $\forall \boldsymbol{u}, \boldsymbol{v}$, $\boldsymbol{u} + \boldsymbol{v}$ is also a vector
- $\exists \mathbf{0}$ such that $\forall \boldsymbol{v}$, $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$
- $\forall \boldsymbol{v}$, there is a vector called $(-\boldsymbol{v})$ such that $\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}$

Any set that has these properties is called a vector space

Examples of Vector Spaces

- The set of real numbers
- The set of all possible forces that can act on a particle
- The set of position vectors of all points in space from a chosen origin
- The set of 3×1 matrices
- The set of 3×3 matrices...in fact, the set of any $m \times n$ matrices
- The set of continuous function on the interval $[0, 1]$.

Basis of Vector Spaces

- Two vectors $\boldsymbol{u}, \boldsymbol{v}$ are said to be linearly independent if $\alpha\boldsymbol{u} + \beta\boldsymbol{v} = \mathbf{0}$ implies that $\alpha, \beta = 0$. Simply put, they are not collinear.
- This idea can be generalised to a collection of vectors
- Given a vector space, how many linearly independent vectors are possible?
- The maximum number of linearly independent vectors is called *dimension*

Basis of Vector Spaces contd...

- A basis $\boldsymbol{a}_i$ is a set of linearly independent vectors such that *every vector* can be written as a linear combination of the basis vectors

$$\boldsymbol{v} = \sum_i v_i \boldsymbol{a}_i$$

where v_i are called the “components” of $\boldsymbol{v}$ with respect to the basis

Scalar Product

For any two vectors $\boldsymbol{u}, \boldsymbol{v}$ in our vector space, we can define a number called the scalar product of $\boldsymbol{u}$ and $\boldsymbol{v}$ (denoted as $\boldsymbol{u} \cdot \boldsymbol{v}$) which satisfies

- $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$
- $\boldsymbol{u} \cdot (\alpha \boldsymbol{v}) = \alpha(\boldsymbol{u} \cdot \boldsymbol{v})$
- $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$
- $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0, \forall \boldsymbol{u} \quad \text{and} \quad \boldsymbol{u} \cdot \boldsymbol{u} = 0 \implies \boldsymbol{u} = \mathbf{0}$
- Two vectors are said to be orthogonal if $\boldsymbol{u} \cdot \boldsymbol{v} = \mathbf{0}$

Scalar Product on Function Spaces

- Scalar product of two "usual vectors" $\boldsymbol{u} \cdot \boldsymbol{v}$ is defined in the "usual way"
- How to define scalar product of two functions?

Scalar Product on Function Spaces

- Well, consider $C[0,1]$, the set of all continuous integrable function defined on $[0,1]$
- If $f, g \in C[0,1]$, then $f \cdot g$ the scalar product of f and g is defined as

$$f \cdot g = \int_0^1 f(x) g(x) dx$$

- Verify that all the properties of scalar product are satisfied

Orthonormal Basis

The scalar product can be used to define an orthonormal basis $\{\boldsymbol{e}_i\}$

- $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$
- Kronecker symbol δ_{ij}

$$\begin{aligned} \delta_{ij} &= 1 \quad i = j \\ \delta_{ij} &= 0 \quad i \neq j \end{aligned}$$

- A vector $\boldsymbol{v}$ can be written in terms of an orthonormal basis

$$\boldsymbol{v} = \sum_i v_i \boldsymbol{e}_i$$

where $v_i = \boldsymbol{v} \cdot \boldsymbol{e}_i$ are the "components" of $\boldsymbol{v}$.

Representations

- Clearly the choice of an orthonormal basis is not unique
- Therefore, there are many possible representations
- Position vectors can also be represented by matrices. For example, a three dimensional vector with respect to a chosen basis $\{\boldsymbol{e}_i\}$ can be represented by a 3×1 matrix $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
- How do we add vectors, multiply by scalars and take scalar products if they are represented as matrices?

Operators

A linear operator (just called operator) is a map $\boldsymbol{A}$ defined on $\mathcal{V}$. For every vector $\boldsymbol{v}$, $\boldsymbol{A}\boldsymbol{v}$ is another vector in $\mathcal{V}$. The operator $\boldsymbol{A}$ satisfies

- $\boldsymbol{A}(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{u}+\boldsymbol{A}\boldsymbol{v}$
- $\boldsymbol{A}(\alpha\boldsymbol{v}) = \alpha\boldsymbol{A}\boldsymbol{v}$
- There is an operator called the identity operator $\boldsymbol{I}$, such that $\boldsymbol{I}\boldsymbol{v} = \boldsymbol{v}$
- There is an operator called $\boldsymbol{O}$ such that $\boldsymbol{O}\boldsymbol{v} = \boldsymbol{0}$

Give examples (physical and geometrical) of operators

Operators contd.

The operator is uniquely determined by the image of the basis vectors

- $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{A}(\sum_i v_i \boldsymbol{e}_i) = \sum_i v_i (\boldsymbol{A}\boldsymbol{e}_i)$
- Let $\boldsymbol{A}\boldsymbol{e}_i = \sum_j A_{ji} \boldsymbol{e}_j$
- Thus the array of number A_{ji} make up a matrix representation of the operator with respect to the basis $\{\boldsymbol{e}_i\}$
- In 2D, if $\boldsymbol{u} = \boldsymbol{A}\boldsymbol{v}$, the matrix representation is

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Product of Operators

- Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are operators, the product $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as

$$\boldsymbol{AB}(\boldsymbol{v}) = \boldsymbol{A}(\boldsymbol{B}(\boldsymbol{v}))$$

- In matrix representation $(AB)_{ij} = \sum_k A_{ik}B_{kj}$
- The inverse $\boldsymbol{A}^{-1}$ of an operator $\boldsymbol{A}$ is defined such that

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

- Operators for which inverses are defined are called invertible operators

Real-Symmetric Operators

- The transpose of an operator $\boldsymbol{A}$, called $\boldsymbol{A}^T$, is such that in any matrix representation $A^T_{ij} = A_{ji}$.
- Real symmetric operators are ubiquitous in physics
- Give examples
- Given a real symmetric operator, we can define a number with every vector $\boldsymbol{v}$ in $\mathcal{V}$ as

  $$\boldsymbol{v} \cdot \boldsymbol{A}\boldsymbol{v}$$

  which is called a "quadratic form"
- Give examples of quadratic forms in physics

Eigenvalues and Eigenvectors

- A *non-zero* vector $\boldsymbol{v}$ is said to the eigenvector of $\boldsymbol{A}$ if

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$$

  where λ is a number
- Physical meaning?
- Given an operator, how to determine its eigenvalues and eigenvectors?
- An example: Find eigenvalues and eigenvectors of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
- Will eigenvalues be always real? How many eigenvectors are there for a general operator?

The Sledgehammer Theorem

- The eigenvalues of a real-symmetric operator are real. The eigenvectors corresponding to distinct eigenvalues are orthogonal!
- Prove this theorem
- Thus, the eigenvectors of a real-symmetric operator can be used as a basis of the vector space!
- If we use the eigenvectors $\boldsymbol{A}$ we call it the $\boldsymbol{A}$ representation. If we use some other operator $\boldsymbol{B}$, we call it he $\boldsymbol{B}$ representation.

Operators on Function Spaces?

- How do we define linear operators on space of functions? Consider the space of infinitely smooth function $C^\infty[0,1]$
- Consider the operator X defined on $C^\infty[0,1]$. Thus, Xf will be a function...

  $$(Xf)(x) = xf(x)!$$

  or take D, such that

  $$(Df)(x) = \frac{df}{dx}!!$$

  These are both linear operator (show this).
- What is the analogue of real symmetric operator, and does the sledge hammer theorem work here?

Absolutely Yes!

- We will not bother to formally define real symmetric operators on function spaces. We will know when we see one!
- Consider the space of infinitely smooth functions defined on $[0, 1]$ which vanish a 0 and 1 – call this $C^{\infty}_{0,0}[0, 1]$ (is this a vector space?)
- Consider the operator D^2 defined on $C^{\infty}_{0,0}[0, 1]$ as

$$(D^2 f)(x) = \frac{d^2 f}{dx^2}(x)$$

- What are its eigenvalues and eigenvectors?

Eigenvectors of D^2 on $C^\infty_{0,0}[0,1]$

- There are infinite eigenvalues and eigenvectors

$$\lambda_n = -n^2\pi^2 \quad \text{and} \quad A_n \sin n\pi x!$$

- Are they orthogonal? (Yes, *of course!!!*)
- We can orthonomalise them (Choose $A_n = \sqrt{2}$)
- And the sledgehammer tells us that any vector in $C^\infty_{0,0}[0,1]$ can be written in the D^2-representation as

$$f(x) = \sum_{n=1}^{\infty} a_n(\sqrt{2}\sin n\pi x)$$

where a_n are the components of f in the D^2-representation with

$$a_n = \int_0^1 f(x)(\sqrt{2}\sin n\pi x)dx$$

Complex Vector Space

A set $\mathcal{V} = \{\boldsymbol{v} | \boldsymbol{v} \text{ is a vector}\}$ is called a complex vector space if there is a binary operation '$+$' (associative) and scalar multiplication which satisfy

- $\forall \boldsymbol{v} \in \mathcal{V}, \alpha \boldsymbol{v} \in \mathcal{V}$ is also a vector $\forall$ *complex numbers* α
- $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v}$ is also a vector
- $\exists \mathbf{0}$ such that $\forall \boldsymbol{v}, \boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$
- $\forall \boldsymbol{v}$, there is a vector called $(-\boldsymbol{v})$ such that $\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}$

Scalar Product in Complex Spaces

For any two vectors $\boldsymbol{u}, \boldsymbol{v}$ in the complex vector space, we can define a number called the scalar product of $\boldsymbol{u}$ and $\boldsymbol{v}$ (denoted as $\boldsymbol{u} \cdot \boldsymbol{v}$) which satisfies

- $\boldsymbol{u} \cdot \boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{u})^*$
- $\boldsymbol{u} \cdot (\alpha \boldsymbol{v}) = \alpha(\boldsymbol{u} \cdot \boldsymbol{v})$
- $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$
- $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0, \forall \boldsymbol{u} \quad \text{and} \quad \boldsymbol{u} \cdot \boldsymbol{u} = 0 \implies \boldsymbol{u} = \mathbf{0}$
- Two vectors are said to be orthogonal if $\boldsymbol{u} \cdot \boldsymbol{v} = \mathbf{0}$
- How do you represent this in matrix notation?

Complex Function Spaces

A possible generalization of $C[0, 1]$ which is the set of all continuous real valued functions defined on the interval $[0, 1]$, is the set of all complex valued continuous functions defined on $[0, 1]$, which scalar multiplication being understood as being that by complex numbers. The scalar product (also called *inner product*) in complex function spaces is defined as

$$f \cdot g = \int_0^1 f^*(x) g(x) dx$$

Operators on Complex Spaces

- Defined precisely as in the case of real spaces
- The idea of a transpose operator is generalized to that of *Hermitian conjugate*
- The Hermitian conjugate $\boldsymbol{A}^\dagger$ of $\boldsymbol{A}$ is defined in matrix notation as $(A^\dagger)_{ij} = A^*_{ji}$
- An operator $\boldsymbol{A}$ is said to be *Hermitian* if $\boldsymbol{A}^\dagger = \boldsymbol{A}$

Operators on Complex Spaces

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- An operator $\boldsymbol{A}$ is said to be *Hermitian* if $\boldsymbol{A}^\dagger = \boldsymbol{A}$

The *Real* Sledgehammer Theorem

- The eigenvalues of a Hermitian operator are real. The eigenvectors corresponding to distinct eigenvalues are orthogonal!
- Prove *this* theorem
- Thus, the eigenvectors of a Hermitian operator can be used as a basis of the complex vector space!
- If we use the eigenvectors $\boldsymbol{A}$ we call it the $\boldsymbol{A}$ representation. If we use some other operator $\boldsymbol{B}$, we call it he $\boldsymbol{B}$ representation
- Modern formulations of quantum mechanics rests on the mathematics of Hermitian operators
- And thats what we move on to!