Solution

1. (20 marks) A person believes that drinking water at the body temperature (37°C) will enable her to live longer. She has with her a measuring jar, a kitchen induction heater, some utensils, but she *does not* have a thermometer. She sees on her mobile phone that the ambient temperature (Bangalore winter) is 20°C. She decides to make body temperature water by taking 500ml of water at ambient temperature and bringing it to boil using the heater. Her idea is to add ambient temperature water to the boiling water immediately after switching off the heater. How much ambient temperature water (in ml) should she add to the boiling water so that the final temperature of the water is her body temperature. Assume that specific heat of water is constant, and that there are no losses.

Mass of boiling water $= M_{h}$
Temperature of boiling water $= T_{h}$
Mass of ambient water $= M_a$
Temperature of ambient water $= T_a$
Body temperature $= T_b$.

By first law of thermodynamics
Heat lost by hot water is heat gained by ambient water

$$M_h c (T_h - T_b) = M_a c (T_b - T_a)$$

where c = specific heat of water

$$\frac{M_a}{M_h} = \frac{T_h - T_b}{T_b - T_a}$$; Denote V for volume

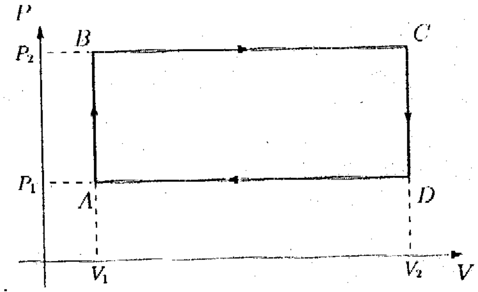
$M_a = \rho_a V_a$ and $M_h = \rho_a V_h$
where ρ_a is the density of ambient water

we get $$V_a = \left(\frac{T_h - T_b}{T_b - T_a}\right) V_h$$

Numerically
$$V_a = \left(\frac{100 - 37}{37 - 20}\right) 500 = 1852.94\text{ml} \simeq 1853\text{ml}$$

Solution

2. (20 marks) A "brilliant" engineer decides to design an engine which uses *one mole of a monoatomic ideal gas*. Being highly imaginative, he comes up with the following cycle $ABCD$:



where $\frac{V_2}{V_1} = \alpha$ and $\frac{P_2}{P_1} = \beta$. Note: $\alpha > 1$ and $\beta > 1$.

(a) Find the heat transferred Q, work W done *on* the engine for each of the segments AB, BC, CD and DA. Express your answers in terms of the gas constant R, the absolute temperature T_A at A, α and β.

(b) Find the total heat that the engineer has to provide to the engine Q_{inp}, and the total work $|W_{eng}|$ done *by* the engine. Find the efficiency $|W_{eng}|/Q_{inp}$.

(c) What is the maximum absolute temperature T_{max} attained in the cycle? Where is it attained?

(d) What is the minimum absolute temperature T_{min} attained in the cycle? Where is it attained?

(e) Use T_{max} and T_{min} to estimate a "Carnot efficiency" of this engine. Compare this "Carnot efficiency" with actual efficiency obtained in part 2b.

Temperature at A, $T_A = \frac{P_1 V_1}{R}$

-do- B, $T_B = \frac{P_2 V_1}{R} = \beta T_A$

-do- C, $T_C = \frac{P_2 V_2}{R} = \alpha\beta T_A$

-do- D, $T_D = \frac{P_1 V_2}{R} = \alpha T_A$.

Since we have an ideal gas as working substance, the change in internal energy in any process $\Delta E = \frac{3}{2} R \Delta T$

a) We use first law of thermodynamics in each of the process

$$Q + W = \Delta E \qquad (1)$$

heat added ← Q; W ↓ Work done on the engine; ΔE ↳ change in internal energy.

As stated $\Delta E = \frac{3}{2} R\Delta T$. We can calculate the work done and obtain Q from (1)

Segment	$\Delta E = \frac{3}{2} R\Delta T$	$W = -\int P dV$	$Q = \Delta E - W$
AB	$\frac{3}{2} R(\beta-1)T_A$	0	$\frac{3}{2} R(\beta-1)T_A$
BC	$\frac{3}{2} R\beta(\alpha-1)T_A$	$-P_2(V_2-V_1)$ $= -P_2 V_1(\alpha-1)$ $= -RT_A\beta(\alpha-1)$	$\frac{5}{2} R\beta(\alpha-1)T_A$
CD	$\frac{3}{2} R\alpha(1-\beta)T_A$	0	$\frac{3}{2} R\alpha(1-\beta)T_A$
DA	$\frac{3}{2} R(1-\alpha)T_A$	$-P_1(V_1-V_2)$ $= RT_A(\alpha-1)$	$\frac{5}{2} R(1-\alpha)T_A$

b) $$Q_{inp} = Q_{AB} + Q_{BC} = \left[\frac{3}{2}(\beta-1) + \frac{5}{2}\beta(\alpha-1)\right]RT_A$$

$$|W_{eng}| = -(W_{BC} + W_{DA}) = (\alpha-1)(\beta-1)RT_A$$

$$\text{Efficiency } \eta = \frac{2(\alpha-1)(\beta-1)}{3(\beta-1) + 5\beta(\alpha-1)}$$

Solution

c) Maximum absolute temperature

$$T_{max} = \alpha\beta T_A$$

occurs at C

d) Minimum absolute temperature

$$T_{min} = T_A$$

occurs at A

e) Estimate of Carnot efficiency

$$\eta_{carnot} = 1 - \frac{T_{min}}{T_{max}} = 1 - \frac{1}{\alpha\beta}$$

It is easy to see that $\forall \alpha, \beta$

$\eta_c \not< \eta$ i.e, η is always smaller than η_c

Simple argument:

Let $\beta > 1$ and $\alpha \to 1$

The. $\eta_c = 1 - \frac{1}{\beta}$ but $\eta = 0$.

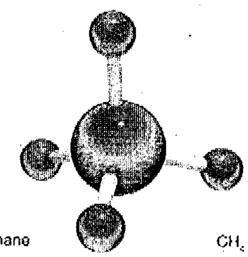
Similarly $\beta \to 1$ and $\alpha > 1$

$\eta_c = 1 - \frac{1}{\alpha}$ but $\eta = 0$.

Solution

Solution

3. (20 marks) A methane molecule is shown below:



(Figure taken from the internet)

Neglecting vibrations of the molecule, find an expression for the specific heat at constant volume of one mole of methane gas using classical mechanics arguments.

Each molecule has three translational degrees of freedom, and three rotational degrees of freedom. The energy of the molecule is

$$\frac{\vec{p}^{\,2}}{2m} + \frac{1}{2}\vec{\omega}\cdot \underset{\sim}{I}\,\vec{\omega}$$

where $\underset{\sim}{I}$ is the momentum of inertia tensor

There are a total of six quaratic degrees of freedom.

By equipartition theorem, each quaratic degree of freedom contributs a molar specific heat of $\frac{1}{2}R$.

Thus the total molar specific heat at constant volume is

$$C_V = 6\cdot\frac{1}{2}R = 3R.$$

Solution

4. (20 marks) Consider a particle of mass m moving in 1-dimensional space experiencing a potential

$$V(x) = A|x|,$$

where A is a positive constant.

(a) Make a schematic plot of the potential $V(x)$. State the dimensions of A.

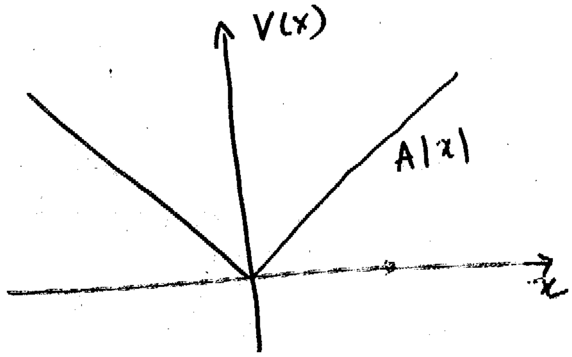
(b) Write down the Hamiltonian of this system.

(c) Given that the particle has energy E, draw a phase portrait of the motion of the particle.

(d) Use Bohr-Sommerfled quantization condition and obtain the discrete energy levels of the system.

(e) Use Heisenberg uncertainty principle to obtain an estimate of the ground state energy of the system. Does it agree with the result of the Bohr-Sommerfeld treatment?

(a)



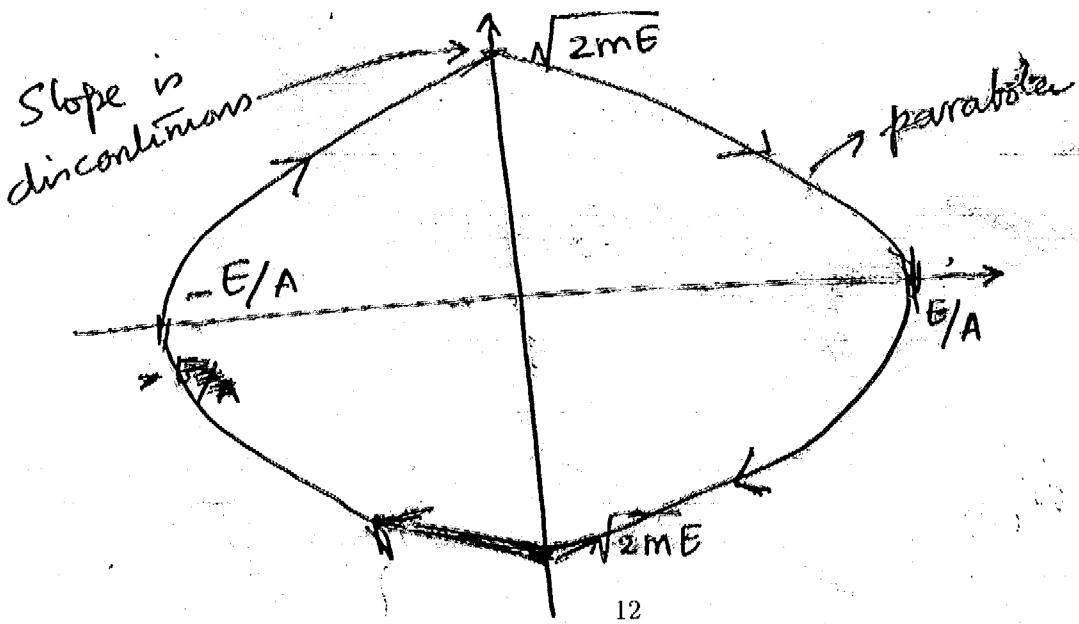
$[A]$ = dimension of force $= MLT^{-2}$

(b) P is momentum

$$H = \frac{p^2}{2m} + V(x) = \frac{p^2}{2m} + A|x|$$

(c) Phase trajectory satisfies

$$\frac{p^2}{2m} + A|x| = E$$



Solution

d) Bohr - Sommerfeld quantization

$$\oint p \cdot dx = nh$$

$\oint dx\, p$ is the area enclosed by the phase trajectory.

We calculate one quarter of the area

$$\frac{\text{Area}}{4} = \int_0^{\sqrt{2mE}} dp \; \frac{1}{A}\left(E - \frac{p^2}{2m}\right) = \frac{1}{A}\left[E\sqrt{2mE} - \frac{1}{6m}(2mE)^{3/2}\right]$$

$$= \frac{1}{A} \; E\sqrt{2mE} \times \left(1 - \frac{1}{3}\right) = \frac{2\sqrt{2m}E^{3/2}}{3A}$$

Thus $\oint p\,dx = \dfrac{8\sqrt{2m}}{3A} E^{3/2}$

one gets

$$E_n = \left(\frac{3Ah}{8\sqrt{2m}}\right)^{2/3} n^{2/3}.$$

e) Let the uncertainty in position be

$\left.\begin{array}{l}\Delta x \approx l \\ \Rightarrow \Delta p \sim \frac{h}{l}\end{array}\right\}$, then $\dfrac{\Delta p^2}{2m} \sim \dfrac{h^2}{2ml^2}$

Thus G.S. Energy $\approx \dfrac{h^2}{2ml^2} + Al$

minimize w.r.t. l to get

$$\frac{h^2}{ml^3} = A \quad \text{or} \quad l^3 \sim \frac{h^2}{mA}.$$

Energy of the ground state

$$\frac{h^2}{2m}\left(\frac{mA}{h^2}\right)^{\frac{2}{3}} + A\left(\frac{h^2}{mA}\right)^{1/3} \approx \frac{3}{2}\left(\frac{Ah}{\sqrt{m}}\right)^{2/3}$$

Consistent with Bohr Sommerfeld

Solution

5. (20 marks, please attempt this question only after you have answered all other questions) One mole of a gas made of "objects" is found experimentally to have a specific of $C_V = \frac{22}{7}R$ (R is the gas constant) at constant volume. The specific heat is found to be temperature independent over the temperature range of measurement. Using classical physics, make a microscopic model of the "object" and demonstrate that your model predicts the measured C_V.

First observe

$$\frac{22}{7} = \frac{21+1}{7} = 3 + \frac{1}{7}. \qquad -(1)$$

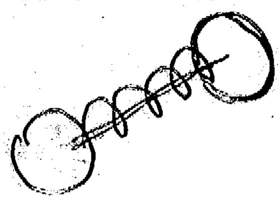
Next observe that

$$3 = \frac{3}{2} + \frac{3}{2} = \frac{3}{2} + 1 + \frac{1}{2}. \qquad -(2)$$

Thus

$$\frac{22}{7} = \underbrace{\frac{3}{2}}_{\text{Translational motion of the object}} + \underbrace{\left(\frac{1}{2}+\frac{1}{2}\right)}_{\text{Rotational motion of the object with two axes}} + \underbrace{\frac{1}{2}+\frac{1}{7}}_{\text{Vibrational contribution.}}$$

The object looks like a dumbell



with a non linear spring

[Solution]

Let p denote the momentum associated with the relative motion of along the axis, and u denote the deformation

Claim : The energy associated with the axial (stretching of the molecule) goes as

$$E = \frac{p^2}{2\mu} + A|u|^7$$

(μ, reduced mass)

momentum is a quadratic degree of freedom $\rightarrow$ so contributes $\frac{1}{2}$ to specific heat.

The "spring" is nonlinear and has an energy that goes as $|u|^7$

$$\langle E_{stretch} \rangle = \frac{\int_{-\infty}^{\infty} du \; A|u|^7 \, e^{-\frac{A|u|^7}{k_B T}}}{\int_{-\infty}^{\infty} du \; e^{-\frac{A|u|^7}{k_B T}}}$$

(using units where k_B ~~=1~~)

(Solution)

Consider

$$Z = \int_{-\infty}^{\infty} du\, e^{-\frac{A|u|^7}{k_B T}} \underset{\beta = 1/k_B T}{=} \int_{-\infty}^{\infty} du\, e^{-A\beta |u|^7}$$

$$= 2\int_0^{\infty} du\, e^{-A\beta |u|^7}$$

define $A\beta u^7 = \xi$

$7 A\beta u^6 du = d\xi$

or

$$du = \frac{1}{7 A\beta}\left(\frac{A\beta}{\xi}\right)^{6/7} d\xi \;.\; \Rightarrow Z = \frac{2}{7}(A\beta)^{-1/7} \underbrace{\int_0^{\infty} du\, u^{-6/7} e^{-u}}_{\Gamma(1/7)}$$

$$Z = \frac{2}{7}(A\beta)^{-1/7}\,\Gamma(1/7)\,.$$

$$\langle E_{stretch}\rangle = -\frac{\partial \ln Z}{\partial \beta} = \frac{1}{7}\frac{1}{\beta} = \frac{k_B T}{7}$$

Thus the average potential energy per molecule at temperature T goes as $\frac{k_B T}{7}$

$$C_v = \frac{k_B}{7} \text{ or molar value } = \frac{R}{7}$$

This is also temperature independent.

Microscopic model of object:

Dumbell shaped molecule whose stretching is nonlinear with a potential energy that goes as $|x|^7$.