1+1 Physics – To Go

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Preface

Here I attempt to put down a stream of conciousness that summarizes what I understand of physics in 1+1, i. e., one space and one time dimensions. The material is freely lifted from various sources without attribution – the best way to lift!

1

Special Relativity

1.1 Einstein's ideas

In early 1900s Einstein concluded that radical changes are necessary in our views about space and time. This was forced on him by the theory of electromagnetism of Maxwell (who was inspired by Faraday, among others) which refused to obey the "rules" of Galilean transformations so stringently verified in "real life" of those times (late 1800s). With no obvious way to tame electromagnetism into the Galilean framework, Einstein decided to abandon it make a new one which would entail reformulating Newtonian mechanics.

Einstein still held steadfast on to a couple of ideas of Galileo. First is that of an *inertial frame* which I shall not venture to define – you know one when you see one. Second, is the

THE PRINCIPLE OF RELATIVITY
Laws of physics are identical in all inertial frames.

Einstein agreed insisted that physics as seen by two people moving uniformly with respect to each other would be identical – this is the first postulate of the special theory of relativity. You might squirm thinking – but isn't this what Newton said – so what is the big deal? The key point realized by Einstein is that to bring mechanics and electromagnetism into a unified theory one needs a *second* postulate.

"INVARIANCE" OF SPEED OF LIGHT
The speed of light c *as measured by all inertial observers is the same.*

There is no "trick" – all observers agree on the units – and the speed of light is always 3×10^8meters/second for all inertial observers! This, at first (and even now for me), is quite puzzling. The natural question that arises is, why in an egalitarian world would light be allowed a special favour? Rather, that being bogged down by this question, we will proceed with exploring the consequences of this postulate.

1.2 Spacetime...or Timespace

To do this we need to introduce the idea of spacetime and we will do this a lá Minkowski. The key concept is that of an *event*. An event is a time t and place in space r. In a world with one spatial dimension, an event is described by an ordered pair (t, r). This, obviously, is not very comforting since t (units of time) and r (which has units of length) have different dimensions, so that natural thing to do is to define an event by the ordered pair (ct, x). We will redefine this as $(x^0, x^1) \equiv (ct, x)$, an act that is equivalent to setting $c = 1$. We will also use index notation x^μ, $\mu = 0, 1$ to denote and event. We see why we say that we are in 1+1 – we are in a world with 1 time and 1 space dimension each.

The set of all events (x^0, x^1) is the 1+1 Minkowski space or the 1+1 timespace (or spacetime). A "path" connecting two events is called a "world line". A world line can describe the trajectory taken by a "particle". If, for example, x^μ_A corresponds to a flashlight being turned on, x^μ_B could be an event where the light from the flashlight reaches. Assuming that the two events are "close together" we define

$$dx^\mu = x^\mu_B - x^\mu_A \tag{1.1}$$

which describes an infinitesimal world line from A to B. We now define

$$ds^2 = (dx^0)^2 - (dx^1)^2 \tag{1.2}$$

which can be written in a nice form as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx_\nu dx^\nu \tag{1.3}$$

where

$$g_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4}$$

We have introduced and used the Einstein summation convention where all the repeated indices (one up and one down) are summed over 0 and 1 timespace coordinate labels. The inverse of the g-matrix (metric tensor) is

$$g^{\mu\nu} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.5}$$

The two gs which we will distinguish by context, can be used to raise and lower indices.

1.3 What do different observers agree on?

Now the same two events A and B can be observed by another observer who is moving uniformly at speed v with respect to us. She will call the events y^μ_A and y^μ_B, and she will also define

$$dy^\mu = y^\mu_B - y^\mu_A \tag{1.6}$$

and define, an analogous,

$$d\tilde{s}^2 = g_{\mu\nu} dy^\mu dy^\nu. \tag{1.7}$$

Clearly, the two of us will not agree on the event coordinates. But is there something that we will agree on?

To find out what this is, use Einstein's second postulate - the speed of light in both our frames is the same. Now imagine A to be the event when a flash of light denoted by x_A in my frame, and y_A in the moving frame. It reached B which I recorded as x_B, and she y_B. What can we say about ds^2 for a worldline of *light*? Note that (going back to "usual things" t, r) $ds^2 = (dx^0)^2 - (dx^1)^2 = c^2dt^2 - dr^2 = 0$, because light moves with the speed of light! Now, the world line seen by the moving observer $d\tilde{s}^2 = (dy^0)^2 - (dy^1)^2 = c^2d\tilde{t}^2 - d\tilde{r}^2 = 0$ also (where $\tilde{t}, \tilde{r}$ are time and space coordinates seen by the moving observer)!– this is again zero by Einstein's postulate that light speed is same in both frames. We conclude that for worldlines of light $ds^2 = d\tilde{s}^2$, which is the way of expressing Einstein's second postulate.

Now what about the world line of some other object such that is not light? Einstein has nothing explicit to say about this, but we can deduce by invoking a couple of key ideas (which perhaps we should elevate to the level of postulates). The first is the homogeneity of timespace, and second is the isotropy of space. For, the event can have whatever x_A, but the holy grail is ds^2; this also applies to the moving frame. We then conclude that

$$d\tilde{s}^2 = \zeta(v)ds^2 \tag{1.8}$$

where $\zeta(v)$ is a yet to be determined function. By isotropy of space $\zeta(v)$ must depend only on the magnitude of v since moving to the left will produce no different physics than moving to the right. Thus $\zeta(v) \equiv \zeta(|v|)$. But now we realize that from the point of view of the moving observer

$$ds^2 = \zeta(-v)d\tilde{s}^2 = \zeta(|v|)d\tilde{s}^2 \tag{1.9}$$

Taking these results together we see that

$$\zeta(|v|)^2 = 1, \tag{1.10}$$

independent of v! Now $\zeta(v = 0) = 1$, and thus we are forced to conclude $\zeta(v) = 1$. The upshot of the discussion is the

INVARIANCE OF ds^2

Einsteins postulate along with homogeneity and isotropy of timespace ensure that ds^2 is invariant between frames.

1.4 The Lorentz Transformation

The next question is what is a transformation rule that relates y^μ with x^μ such that ds^2 is invariant for every set of two events A and B in both the frames. It is natural to look for a transformation of the form

$$y^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{1.11}$$

What are Λ^μ_ν and a^μ? Note that dy^μ does not depend on a^μ. This is because a^μ stands for a shift of the the origin of timespace. In other words, what I would call an event at my spatial and temporal origin will be called as a^μ by my friend in the moving frame. It is also easy to see that Λ^μ_ν depends on the velocity v of the my friend's frame. This follows from

$$\begin{aligned} d\tilde{s}^2 = g_{\mu\nu} dy^\mu dy^\nu &= g_{\mu\nu} \Lambda^\mu_\sigma \Lambda^\nu_\eta dx^\sigma dx^\eta \\ &= g_{\sigma\eta} dx^\sigma dx^\eta = ds^2. \end{aligned} \tag{1.12}$$

Since ds^2 and $d\tilde{s}^2$ are related by a factor $\zeta(v)$, it follows that Λ depends on v. The last equation gives

$$\Lambda^\mu_\sigma g_{\mu\nu} \Lambda^\nu_\eta = g_{\sigma\eta} \tag{1.13}$$

From this, one can go on to show that the Λ matrix must have the form

$$\Lambda \equiv \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \tag{1.14}$$

and evidently θ depends on v. How do we determine this dependence? The worldline of the moving observer as seen by me is given by the equation

$$a^1 = \Lambda^1_\nu x^\nu + a^1, \tag{1.15}$$

this because, in her reference frame, her position coordinate is always a^1. Writing this in "usual quantities" gives us

$$0 = \sinh\theta \times (ct) + \cosh\theta \times r \tag{1.16}$$

which is the equation of the trajectory of the moving observer in our frame. We immediately have

$$\tanh\theta = -\frac{v}{c} \equiv \beta \tag{1.17}$$

and

$$\Lambda \equiv \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \tag{1.18}$$

Incidently, the form in eqn. (1.14) is not the only way we can write this; for example, the $\cosh\theta$ could have had a $-$ sign in front of it. We will not bother about such nicities for now, and stick to the from in eqn. (1.14) which defines what is called an

ORTHOCHRONOUS LORENTZ(POINCARÉ) TRANSFORMATION

$$\begin{pmatrix} y^0 \\ y^1 \end{pmatrix} = \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} + \begin{pmatrix} a^0 \\ a^1 \end{pmatrix} \tag{1.19}$$

1.5 Some Consequences

1.5.1 "Simultaneity" is *not* invariant!

Two events are said to be simultaneous in a frame, if they both have the same time ("0") coordinate in that frame. A key consequence of the Lorentz transformation is that events that are simultaneous to me are generally not so to my moving friend. This is easily seen by setting $x_A \equiv (0, x^1_A)$ and $x_B \equiv (0, x^1_B)$. The Lorentz transformation shows that y_A ad y_B will have will different time coordinates if $v \neq 0$. A nice way to visualize this is using Einstein's train car analogy.

1.5.2 Time dialation

I am observing my friend move away from me. In her frame, her position coordinate does not change (as embodied in eqn. (1.15)). However, her clock ticks! We would now like to relate an interval $\tilde{t}$ on her clock and the corresponding interval t on my clock. The equation to look at is (in natural coordinates)

$$c\tilde{t} + a^0 = \frac{ct - \beta x}{\sqrt{1-\beta^2}} + a^0 \tag{1.20}$$

Along this world line in our frame $x = vt$, and this

$$\tilde{t} = \sqrt{1-\beta^2} \times t \tag{1.21}$$

Thus time elapsed on the the moving clock is "smaller", i.e., time "runs at a slower rate" on a moving clock – time dialates!

1.5.3 Length contraction

A key point is the process length measurement in a frame of an object like a rod orresponds to two simultaneous events in that frame. Consider our friend carrying a rod AB of length L_0 in her frame. The measurement process gives her two events in her frame say $y_A = (0, 0)$ and $y_B = (0, L_0)$. To simplify the discussion, I have taken $a^\mu = 0$. However, as we know these events will not be simultaneous for me. By choice of a^μ, we know that $x_A = (0, 0)$ i.e, the A-end of the rod is at my timespace origin at time $t = 0$ in my frame. What I need to do is to find where the other end B of the rod is at time $t = 0$ in my frame, let us denote this by the event $(0, L)$ in my frame. Now the event y_B corresponds to

$$x_B^0 = \gamma\beta L_0, \quad x_B^1 = \gamma L_0 \tag{1.22}$$

But note that the B end of the rod also moves with velocity v respect to me, which means that $x_B^1 = L + \beta x_B^0$. The upshot is that

$$L = \sqrt{1 - \beta^2} \times L_0 \tag{1.23}$$

This is the "length contraction", an moving object appears shorter in its direction of motion.

1.5.4 Einstein's Law of Velocity Addition

Suppose I observe a particle move in my frame and describe this by a worldline $x^\mu(\tau)$ where τ is a parameter (we will see later that there is natural choice for this parameter). My friend who is moving with a velocity v with respect to me describes the same phenomenon by a world line $y^\mu(\tau)$. The velocity of the particle measured in each of the frames

$$w_x = c\frac{dx^1}{dx^0}. \quad w_y = c\frac{dy^1}{dy^0} \tag{1.24}$$

and the natural question is how is w_y related to w_x (the Galilean answer is $w_y = w_x - v$). By Lorentz transformation

$$w_y = c\left(\frac{-\beta dx^0 + dx^1}{dx^0 - \beta dx^1}\right) = \frac{w_x - v}{1 - \frac{v w_x}{c^2}} \tag{1.25}$$

which is the Einstein's law of velocity addition. It has all the expected features – reduces to Galilean result for $\beta \ll 1$ and we get $w_y = c$ if $w_x = c$ (which is Einstein's second postulate).

1.5.5 Proper velocity

It is rather disconcerting that the velocity addition law eqn. (1.25) does not have a "nice" form, in the sense that the velocity addition law is "nonlinear" unlike the Lonrentz transformation. Also, there is another philosophical point – the Lorentz transformation treats space and time on an "equal" footing, while the velocity addition law speaks only about velocity of the type $\frac{dr}{dt}$.

Here is where the parameter τ introduced at the beginning of the last section comes to our rescue. Let us define something which is called a *proper velocity* in my frame as

$$u_x^\mu = \frac{dx^\mu}{d\tau} \tag{1.26}$$

and my moving friend will define it as

$$u_y^\mu = \frac{dy^\mu}{d\tau} \tag{1.27}$$

Here is a beautiful thing: if we choose τ as the *proper time associated with the particle* we both are observing, then some nice things happen. In fact, since the particle could be moving non-uniformly, the natural thing to look at is not τ itself, but a differential $d\tau$. Clearly,

$$d\tau = \sqrt{1 - \left(\frac{w_x}{c}\right)^2} \times dt = \sqrt{1 - \left(\frac{w_y}{c}\right)^2} \times d\tilde{t} \quad (1.28)$$

Since $d\tau$ is time seen by the particle in *its* frame, this does not change when we go from my frame to my friend's frame. We know that $dy^\mu = \Lambda^\mu_\nu dx^\nu$, and this immediately gives

$$u^\mu_y = \Lambda^\mu_\nu u^\nu_x \quad (1.29)$$

Ah, the proper velocity transforms nicely! And, in fact, we see that

$$g_{\mu\nu} u^\mu_y u^\nu_y = g_{\mu\nu} u^\mu_x u^\nu_x = c^2 \quad (1.30)$$

If we were to write in "usual language", we see that

$$\begin{pmatrix} u^0_x \\ u^1_x \end{pmatrix} = \frac{1}{\sqrt{1-\beta_x^2}} \begin{pmatrix} c \\ w_x \end{pmatrix} \quad (1.31)$$

where $\beta_x = \frac{w_x}{c}$.

1.5.6 Doppler Effect

Light is described by a wave, i. e., using "usual language" by

$$\phi(t, x) = Ae^{(i(kr-\omega t))} \tag{1.32}$$

where where k is the wavevector, and $\omega = ck$ is the frequency. This wave propagates (in my frame) towards the positive r-axis in my frame. This can be written in a nicer form for the present purposes

$$\phi(x^\mu) = Ae^{-ik_\mu x^\mu} \tag{1.33}$$

where

$$\begin{pmatrix} k^0 \\ k^1 \end{pmatrix} = \begin{pmatrix} \frac{\omega}{c} \\ k \end{pmatrix} \tag{1.34}$$

and, of course, $k_\mu = g_{\mu\nu}k^\nu$. Now in my friends moving frame (moving with velocity v with respect to me) the same light wave is described by $\tilde{\phi}(y^\mu)$, and it is natural to demand

$$\tilde{\phi}(y^\mu) = \phi(x^\mu), \tag{1.35}$$

and it is immediate that

$$\tilde{\phi}(y^\mu) = Ae^{-i\tilde{k}_\mu y^\mu} \tag{1.36}$$

where

$$\tilde{k}_\mu = k_\nu (\Lambda^{-1})^\nu_\mu = \gamma \begin{pmatrix} \gamma(1-\beta)\frac{\omega}{c} & -\gamma(1-\beta)k \end{pmatrix} \tag{1.37}$$

whence we see that

$$\tilde{\omega} = \sqrt{\frac{1-\beta}{1+\beta}} \times \omega \tag{1.38}$$

with a similar formula for $\tilde{k}$. Thus is my friend is moving in the same direction as the light wave (towards $+r$ axis, i.e., $\beta > 0$) then she will see a *red-shifted* light.

This result derived from rather opaque considerations can be seen in a better fashion as follows. Imagine that I have a light-gun which shoots massless bullets (aka photon) at $r = 0$; the light-gun shoots one bullet every $T = \frac{2\pi}{\omega}$ seconds. Now my friend is moving with respect to me at velocity v, and things are set up such that she and I were at $r = 0$ and $t = 0$ (in both our frames) when the light gun emitted its first bullet. Now she continues to move to right, and after a time T in my frame the light gun emits second bullet, which will reach her at time $\tilde{T}$ in *her* frame – she denotes this event as $(0, \tilde{T})$. Question is what is the relationship between $\tilde{T}$ and T? The event of the second light-bullet reaching my friend is observed by me as (ct, vt), where t is the elapsed time in my frame. But we know that the second light-bullet started off from the origin only at a time T later, and thus $c(t - T) = vt$, or $t = \frac{1}{1-\beta}T$. But we know from time dialation formula that

$$\tilde{T} = \sqrt{1 - \beta^2} \times t = \sqrt{\frac{1 + \beta}{1 - \beta}} \times T \tag{1.39}$$

which is precisely eqn. (1.38).

This brings us to the end of a survey of the essential kinematic effects that are brought about by Einstein's postulates.

1.6 Relativistic Dynamics

Newton's law states that $F = \frac{dp}{dt}$, where F is the force and p is the momentum. This is, of course, Galilean invariant. It is Lorentz invariant? If we probe this, we will find soon that there are many difficulties. Let us illustrate this with a simple example. Consider, in my frame two particles of equal mass m moving in opposite directions with velocities w_x and $-w_x$. The collide with each other and come to rest (evidently, a particle of mass $2m$). This entire process conserves momentum (as it should) in my frame. Let us now as my friend moving with velocity v as to what she finds. First of all, the initial total momentum of the two particle in her system is $\frac{m(w_x - v)}{1 - \frac{w_x v}{c^2}} - \frac{m(w_x + v)}{1 + \frac{w_x v}{c^2}}$ (by Einsten's law of velocity addition). After the collision, she will find the momentum to be $-2mv$. In other words she find that momentum is NOT conserved! This is quite unsettling!

Actually, the root cause of all this is the velocity addition law which "does not look nice", and we inherit the "problems of velocity" by defining $p = mu$. What would be a natural way to defined momentum? Well we have already seen that the proper velocity transforms nicely, and so it is natural to define momentum as

$$p^\mu = mu^\mu \tag{1.40}$$

It is natural to now insist first that the total p^μ be a conserved quantity. Let us see the consequences of this definition before we go on.

First, it is evident that with this definition the sum total p^1 of the colliding balls in both the frames before and after collision to be equal! So this definition makes law of conservation of of the momentum a "physical law", i. e., something that is same in all frames. But note that what we usually call momentum is only the "1" component of p^μ. And what is more a Lorentz transformation from one frame to another "mixes up" the ps of our frame to give the ps in the other frame. The natural thing to ask for is that the total p^μ vector is conserved. Now this leads a bit of a puzzle with the 0 component. In my frame, the total p^0 before collision was $\frac{2mc}{\sqrt{1-\left(\frac{w_x}{c}\right)^2}}$, this has to be equal to it after collision. What this means is that after the particles have coalesce into one, $p^0 = \frac{2mc}{\sqrt{1-\left(\frac{w_x}{c}\right)^2}}$. Since the velocity of the final particle, one would have expected this to be $2mc$, but quite interestingly we find something larger. Natural question is how do we interpret p^0?

Lets simply begin by trying to interpret the definition. We see that for a particle of mass m

$$p^0 = \frac{mc}{\sqrt{1 - \left(\frac{w_x}{c}\right)^2}} \tag{1.41}$$

We see that this will be

$$p^0 \approx \frac{1}{c}\left(mc^2 + \frac{1}{2}mw_x^2\right) \tag{1.42}$$

be when $|w_x| \ll c$. This apart from the factor of $1/c$ and the constant term mc^2 looks like the (Newtonian, nonrelativistic) kinetic energy of the particle. One suspects then that $p^0 = \frac{E}{c}$!

If this is true, then one should be able to see that rate of work done on the particle by a force F must the rate of change of energy. Consider now a single particle on which a force F is acting. Since p^1 is a consistent definition of momentum, a reasonable form Newton's law is (which reduces to the usual case when $|w_x| \ll c$ is

$$F = \frac{dp^1}{dt} = \frac{d}{dt}\left(\frac{mw_x}{\sqrt{1 - \left(\frac{w_x}{c}\right)^2}}\right) \tag{1.43}$$

Now the rate of work done, or power is

$$
\begin{aligned}
Fw_x &= w_x \frac{d}{dt}\left(\frac{mw_x}{\sqrt{1-\left(\frac{w_x}{c}\right)^2}}\right) \\
&= \frac{mw_x}{(1-\left(\frac{w_x}{c}\right)^2)^{3/2}}\frac{dw_x}{dt} \\
&= c\frac{dp^0}{dt}
\end{aligned}
\tag{1.44}
$$

from where we obtain the relation

$$p^0 = \frac{E}{c} \tag{1.45}$$

where E is the energy of the particle. For a particle of mass m at rest in our frame, $p^0 = mc$ leading to the famous

$$E = mc^2 \tag{1.46}$$

which is the rest energy a particle of mass m. Note that

$$p_\mu p^\mu = \left(\frac{E}{c}\right)^2 - p^2 = m^2c^2 \tag{1.47}$$

(using eqn. (1.30)) which results in

$$E^2 = p^2c^2 + m^2c^4 \tag{1.48}$$