PH 6297: Classical Field Theory
Homework Set 1

August 21, 2017

1. In class we reviewed Lorentz boost transformations in the case when one inertial frame (S') was moving wrt to the another, S along the common x-axis:

$$\begin{aligned} x'^1 &= \gamma\left(x^1 - \beta x^0\right), \\ x'^{(2,3)} &= x^{(2,3)}, \\ x'^0 &= \gamma\left(x^0 - \beta x^1\right). \end{aligned}$$

A. Show that one can generalize these to derive the form of the Lorentz transformations for the case when S' is moving away from S with a velocity not along x-axis, but in an arbitrary direction, i.e. $\mathbf{V} = V^1\hat{\mathbf{e}}_1 + V^2\hat{\mathbf{e}}_2 + V^3\hat{\mathbf{e}}_3$ by completing the hints provided in class (or independent of that):

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} + \frac{\gamma - 1}{\boldsymbol{\beta}^2}\left(\mathbf{x}\cdot\boldsymbol{\beta}\right)\boldsymbol{\beta} - \gamma\boldsymbol{\beta}x^0, \\ x'^0 &= \gamma\left(x^0 - \boldsymbol{\beta}\cdot\mathbf{x}\right), \end{aligned}$$

where, $\boldsymbol{\beta} \equiv \mathbf{V}/c$ and $\boldsymbol{\beta}^2 \equiv \boldsymbol{\beta}\cdot\boldsymbol{\beta}$.

B. Write the above pair of equations first in component form, i.e. $x'^i = x^i + \ldots$, $x'^0 = \gamma x^0 + \ldots$ and then in matrix form,

$$x' = \Lambda x$$

indicating clearly the entries, $\Lambda^\mu{}_\nu$ in terms of γ, β^i.

Solution

A. In class we saw how to generalize the expression for Lorentz transformation from one inertial frame, S to another frame, S'moving with a velocity $\mathbf{v} = v\hat{\mathbf{e}}_1$ along the common x-axis to the case when S' is moving with a velocity, $\mathbf{v} = v_i\,\hat{\mathbf{e}}_i$, in an arbitrary direction by using the following recipe,

$$\begin{aligned} x'^0 &= \gamma\left(x^0 - \beta x^{||}\right), \\ x'^{||} &= \gamma\left(x^{||} - \beta x^0\right), \\ \mathbf{x}'^{\perp} &= \mathbf{x}^{\perp}, \end{aligned}$$

where $x^{||} \equiv \mathbf{x}\cdot\hat{\boldsymbol{\beta}}$, $x'^{||} = \mathbf{x}'\cdot\hat{\boldsymbol{\beta}}$ denote the component of position vector along the/ parallel to the velocity direction, while $\mathbf{x}^{\perp} \equiv \mathbf{x} - \left(\mathbf{x}\cdot\hat{\boldsymbol{\beta}}\right)\hat{\boldsymbol{\beta}}$ and $\mathbf{x}'^{\perp} = \mathbf{x}' - \left(\mathbf{x}'\cdot\hat{\boldsymbol{\beta}}\right)\hat{\boldsymbol{\beta}}$ denote the part of the position vector perpendicular to the velocity, $\boldsymbol{\beta} = \frac{\mathbf{v}}{c}$. Thus we have, the second equation,

$$\begin{aligned} x'^{||} &= \gamma\left(x^{||} - \beta x^0\right) \\ \Longrightarrow \mathbf{x}'\cdot\hat{\boldsymbol{\beta}} &= \gamma\left(\mathbf{x}\cdot\hat{\boldsymbol{\beta}} - \beta x^0\right). \end{aligned}$$

and then substitute this in the third equation (see underlined term in the following),

$$\begin{aligned} \mathbf{x}'^{\perp} &= \mathbf{x}^{\perp} \\ \Longrightarrow \mathbf{x}' - \underline{\left(\mathbf{x}'\cdot\hat{\boldsymbol{\beta}}\right)\hat{\boldsymbol{\beta}}} &= \mathbf{x} - \left(\mathbf{x}\cdot\hat{\boldsymbol{\beta}}\right)\hat{\boldsymbol{\beta}} \\ \Longrightarrow \mathbf{x}' - \gamma\left(\mathbf{x}\cdot\hat{\boldsymbol{\beta}} - \beta x^0\right)\hat{\boldsymbol{\beta}} &= \mathbf{x} - \left(\mathbf{x}\cdot\hat{\boldsymbol{\beta}}\right)\hat{\boldsymbol{\beta}} \\ \Longrightarrow \mathbf{x}' &= \mathbf{x} + (\gamma - 1)\left(\mathbf{x}\cdot\hat{\boldsymbol{\beta}}\right)\hat{\boldsymbol{\beta}} - \gamma\boldsymbol{\beta}x^0 \\ &= \mathbf{x} + \frac{\gamma - 1}{\beta^2}\left(\mathbf{x}\cdot\boldsymbol{\beta}\right)\boldsymbol{\beta} - \gamma\boldsymbol{\beta}x^0. \end{aligned} \tag{1}$$

The first equation too can be rewritten as,

$$\begin{aligned} x'^0 &= \gamma\left(x^0 - \beta x^{||}\right) \\ \Longrightarrow x'^0 &= \gamma\left(x^0 - \mathbf{x}\cdot\boldsymbol{\beta}\right). \end{aligned} \tag{2}$$

B. The matrix form of (1) and (2) can be obtained from their component form. The i-th component of (1)

$$\begin{aligned} x'^i &= x^i + \frac{\gamma - 1}{\beta^2}x^j\beta^j\beta^i - \gamma\beta^i x^0 \\ &= \left(\delta^i_j + \frac{\gamma - 1}{\beta^2}\beta^i\beta^j\right)x^j - \gamma\beta^i x^0 \end{aligned}$$

This gives, $\Lambda^i{}_0 = -\gamma\beta^i$, $\Lambda^i{}_j = \delta^i_j + \frac{\gamma-1}{\beta^2}\beta^i\beta^j$. Similarly, the component form of (2) gives,

$$x'^0 = \gamma x^0 - \gamma\beta^i x^i,$$

i.e., $\Lambda^0{}_0 = \gamma$, and $\Lambda^0{}_i = -\gamma\beta^i$. Now we can write down Λ in matrix form,

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta^1 & -\gamma\beta^2 & -\gamma\beta^3 \\ -\gamma\beta^1 & 1+\frac{\gamma-1}{\beta^2}\left(\beta^1\right)^2 & \frac{\gamma-1}{\beta^2}\beta^1\beta^2 & \frac{\gamma-1}{\beta^2}\beta^1\beta^3 \\ -\gamma\beta^2 & \frac{\gamma-1}{\beta^2}\beta^2\beta^1 & 1+\frac{\gamma-1}{\beta^2}\left(\beta^2\right)^2 & \frac{\gamma-1}{\beta^2}\beta^2\beta^3 \\ -\gamma\beta^3 & \frac{\gamma-1}{\beta^2}\beta^3\beta^1 & \frac{\gamma-1}{\beta^2}\beta^3\beta^2 & 1+\frac{\gamma-1}{\beta^2}\left(\beta^3\right)^2 \end{pmatrix}.$$

2. From the defining equation for a Lorentz transformation,

$$\Lambda^T \eta \Lambda = \eta \tag{3}$$

show that the 00-component (top left corner entry) of this equation gives,

$$\left(\Lambda^0{}_0\right)^2 \geq 1,$$

i.e., either,

$$\Lambda^0{}_0 \geq 1,$$

or,

$$\Lambda^0{}_0 \leq -1.$$

Solution:
Eq. (3) in matrix form can be expressed as,

$$\Lambda^\alpha{}_\mu \, \eta_{\alpha\beta} \, \Lambda^\beta{}_\nu = \eta_{\mu\nu}.$$

The $\mu = \nu = 0$ component of this equation is,

$$\begin{aligned} \eta_{00} &= \Lambda^\alpha{}_0 \, \eta_{\alpha\beta} \, \Lambda^\beta{}_0 \\ \Longrightarrow 1 &= \Lambda^0{}_0 \, \eta_{00} \, \Lambda^0{}_0 + \Lambda^i{}_0 \, \eta_{ii} \, \Lambda^i{}_0 \\ \Longrightarrow 1 &= \left(\Lambda^0{}_0\right)^2 - \Lambda^i{}_0 \Lambda^i{}_0 \end{aligned}$$

or,

$$\begin{aligned} \left(\Lambda^0{}_0\right)^2 &= 1 + \Lambda^i{}_0 \Lambda^i{}_0 \\ \Longrightarrow \Lambda^0{}_0 = \sqrt{1 + \Lambda^i{}_0 \Lambda^i{}_0} \quad &\text{or,} \quad \Lambda^0{}_0 = -\sqrt{1 + \Lambda^i{}_0 \Lambda^i{}_0} \\ \Longrightarrow \Lambda^0{}_0 \geq 1 \quad &\text{or,} \quad \Lambda^0{}_0 \leq -1, \end{aligned}$$

since, $\Lambda^i{}_0 \Lambda^i{}_0 = \left(\Lambda^1{}_0\right)^2 + \left(\Lambda^2{}_0\right)^2 + \left(\Lambda^3{}_0\right)^2 \geq 0$ as it is a sum of squares which has to be positive semidefinite.