PH 6297: Classical Field Theory
Homework Set 2

1. In the last HW set you were asked to find the expression for the Lorentz transformation between two frames, say S and S', such that their respective spatial coordinate axes are parallel, but S' is moving away from S at a velocity $\mathbf{V}$ in a general direction (see fig 1).

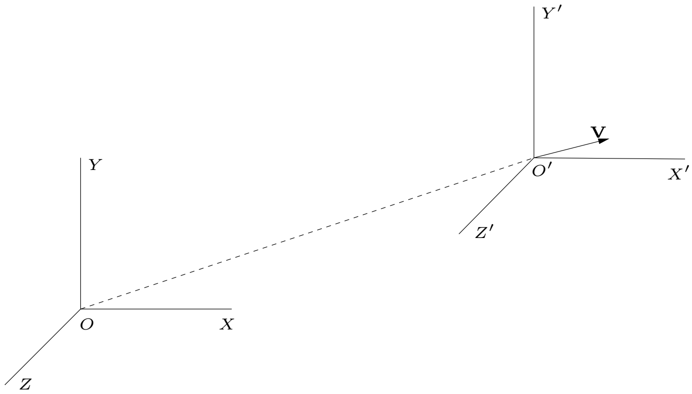

Figure 1: Boost in a general direction, while the coordinate axes are parallel.

Now, in this problem, find the expression of the most general Lorentz transformation, where the frame S' is not only boosted wrt S in a general velocity direction, $\mathbf{V}$, but also the coordinate axes are not parallel anymore. Instead the axes of S' are rotated wrt those of S by an arbitrary angle θ around an arbitrary axis of rotation passing through the origin O'. This axis of rotation is given by the unit vector $\hat{\mathbf{n}}$ [1]. See fig 2

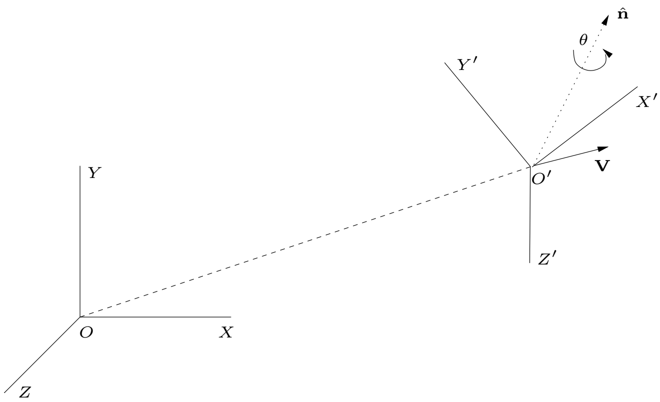

Figure 2: Boost in a general direction, and the coordinate axes are rotated wrt to each other.

[1] whose components are the direction cosines, i.e. $\hat{\mathbf{n}} = \left(\cos\phi, \cos\psi, \sqrt{1-\cos^2\phi-\cos^2\psi}\right)$

Solution:

In this problem, the frame S' is moving away from the frame S with a velocity, $\boldsymbol{\beta}$ and the frame S' is also rotated wrt to S by an angle θ around an axis given by the unit vector, $\hat{\mathbf{n}}$. We will accomplish the Lorentz transformations between the frame S and S' in the following two steps. First we will switch to an intermediate frame, S'', which is rotated wrt to S by an angel θ about an axis given by the unit vector, $\hat{\mathbf{n}} = (n^1, n^2, n^3)$, where n^i's are the direction cosines of the axis vector by an orthogonal transformation, $O(\hat{\mathbf{n}}, \theta)$. Then we will apply a boost from S'' to S' by the rotated velocity $\boldsymbol{\beta}'' \equiv O\boldsymbol{\beta}$.

- Step 1: Rotation
  Recall that under rotations the components of the spatial/3-vector which are parallel to the axis of rotation do not change, only those components which lie in a plane perpendicular to the axis of rotation change in some way. For example for a rotation by an amount θ around the z-axis, the transformations are:

$$\begin{aligned} z'' &= z, \\ x'' &= \cos\theta\, x + \sin\theta\, y, \\ y'' &= -\sin\theta\, x + \cos\theta\, y. \end{aligned}$$

  These can be easily generalized to the case of an arbitray axis, $\hat{\mathbf{n}}$,

$$\begin{aligned} x''_{\parallel} &= x_{\parallel}, \\ \left(x''_{\perp}\right)^i &= \cos\theta\ \left(x_{\perp}\right)^i + \sin\theta\ \left(\mathbf{x}\times\hat{\mathbf{n}}\right)^i, \end{aligned}$$

  where, $x_{\parallel} = \mathbf{x}\cdot\hat{\mathbf{n}}$, $x''_{\parallel} = \mathbf{x}''\cdot\hat{\mathbf{n}}$ are the components parallel to the axis, while, $\mathbf{x}_{\perp} = \mathbf{x} - (\mathbf{x}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}}$ and $\mathbf{x}''_{\perp} = \mathbf{x}'' - (\mathbf{x}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}}$ are parts of the vector $\mathbf{x}$ and $\mathbf{x}''$ respectively perpendicular to the axis. The superscript i represents the i-th component. Rearranging stuff, we can rewrite the above pair in the form,

$$\left(x''\right)^i = \cos\theta\, x^i + (1-\cos\theta)\, n^i n^j\, x^j + \left(\epsilon^{ijk} n^k \sin\theta\right) x^j. \tag{1}$$

  The time-component of course does not transform under spatial rotations, so we have,

$$\left(x''\right)^0 = x^0. \tag{2}$$

  Also, the velocity vector with which the frame S' is moving will now look rotated in the S",

$$\left(\beta''\right)^i = \cos\theta\, \beta^i + (1-\cos\theta)\, n^i n^j\, \beta^j + \left(\epsilon^{ijk} n^k \sin\theta\right) \beta^j. \tag{3}$$

  However the magnitude will remain same,

$$\beta'' = \beta \tag{4}$$

  as well as the scalar product will remain same,

$$\mathbf{x}''\cdot\boldsymbol{\beta}'' = \mathbf{x}\cdot\boldsymbol{\beta}. \tag{5}$$

- Step 2: Boost
  Now we apply a boost by a velocity, $\boldsymbol{\beta}''$ to go from, S'' to the frame, S'. We already know how the Lorentz transformations for this looks like,

$$\begin{aligned}
\mathbf{x}' &= \mathbf{x}'' + \frac{(\gamma''-1)}{(\beta'')^2}\left(\mathbf{x}''\cdot\boldsymbol{\beta}''\right)\boldsymbol{\beta}'' - \gamma''\boldsymbol{\beta}''\left(x''\right)^0, \\
(x')^0 &= \gamma''\left(x''\right)^0 - \gamma''\boldsymbol{\beta}''\cdot\mathbf{x}'',
\end{aligned} \tag{6}$$

  where $\gamma'' = 1/\sqrt{1-(\beta'')^2}$.

Now we combine the result of both transformations to get the full transformation from S to S'. To this end we use Eq.s (2, (4), and, (5), to rewrite the above pair of equations (6),

$$\begin{aligned}
\mathbf{x}' &= \mathbf{x}'' + \frac{(\gamma-1)}{\beta^2}\left(\mathbf{x}\cdot\boldsymbol{\beta}\right)\boldsymbol{\beta}'' - \gamma\boldsymbol{\beta}''x^0, \\
(x')^0 &= \gamma x^0 - \gamma\boldsymbol{\beta}\cdot\mathbf{x}.
\end{aligned}$$

The time component equation immediately gives,

$$\Lambda^0{}_0 = \gamma, \qquad \Lambda^0{}_i = -\gamma\beta^i. \tag{7}$$

The first i.e. spatial vector equation in component form looks like,

$$(x')^i = (x'')^i + \frac{(\gamma-1)}{\beta^2}\left(\beta''\right)^i\beta^j\,x^j - \gamma\left(\beta''\right)^i x^0,$$

and after substituting Eq. (1) looks like,

$$(x')^i = \cos\theta\,x^i + (1-\cos\theta)\,n^in^j\,x^j + \left(\epsilon^{ijk}n^k\sin\theta\right)x^j + \frac{(\gamma-1)}{\beta^2}\left(\beta''\right)^i\beta^j\,x^j - \gamma\left(\beta''\right)^i x^0.$$

This gives,

$$\begin{aligned}
\Lambda^i{}_0 &= -\gamma\left(\beta''\right)^i \\
\Lambda^i{}_j &= \cos\theta\,\delta^i_j + (1-\cos\theta)\,n^in^j + \epsilon^{ijk}n^k\sin\theta + \frac{(\gamma-1)}{\beta^2}\left(\beta''\right)^i\beta^j
\end{aligned} \tag{8}$$

where $\left(\beta''\right)^i$ are defined in (3) . Thus we have the final form of the Lorentz transformations between S and S' in equations (7) and (8).

2. Check that the product of a vector and a 1-form with their indices contracted is a scalar (invariant)under Lorentz Transformation,

$$\omega_\mu a^\mu \rightarrow \omega^{'}_\mu a'^\mu = \omega_\mu a^\mu(9)$$

Solution:

The transformation law for vectors and 1-forms are respectively,

$$\begin{aligned} a^\mu \to a'^\mu &= \Lambda^\mu{}_\nu \, a^\nu \\ \omega_\mu \to \omega'^\mu &= \Lambda_\mu{}^\rho \, \omega_\rho \end{aligned}$$

where we note that by definition, $\Lambda_\mu{}^\nu = \left(\Lambda^{-1}\right)^\nu{}_\mu$. The the product of such a vector and an one-form with their indices contracted transforms like,

$$\begin{aligned} \omega_\mu a^\mu \to \omega'{}_\mu \, a'^\mu &= \Lambda_\mu{}^\rho \, \omega_\rho \, \Lambda^\mu{}_\nu \, a^\nu \\ &= \left(\Lambda^{-1}\right)^\rho{}_\mu \Lambda^\mu{}_\nu \, \omega_\rho a^\nu \\ &= \left(\Lambda^{-1} \, \Lambda\right)^\rho{}_\nu \, \omega_\rho a^\nu \\ &= \delta^\rho_\nu \, \omega_\rho \, a^\nu \\ &= \omega_\nu a^\nu . \end{aligned}$$

So we see when we contract upstairs and downstairs indices the product becomes a Lorentz scalar, i.e., it remains invariant under Lorentz transformations.

3. Prove that Minkowski metric, $\eta_{\mu\nu}$ is a $(0,2)$-rank invariant tensor,

$$\eta_{\mu\nu} \to \eta'{}_{\mu\nu} \equiv \Lambda_\mu{}^\alpha \, \Lambda_\nu{}^\beta \, \eta_{\alpha\beta} = \eta_{\mu\nu} \tag{10}$$

Solution:

Since $\eta_{\mu\nu}$ has two downstairs index, it could represent components of a candidate $(0,2)$ tensor. In which case, the transformation rule under Lorentz transformations would look like,

$$\begin{aligned} \eta_{\mu\nu} \to \eta'{}_{\mu\nu} &\equiv \Lambda_\mu{}^\alpha \, \Lambda_\nu{}^\beta \, \eta_{\alpha\beta} \\ &= \left(\Lambda^{-1}\right)^\alpha{}_\mu \, \eta_{\alpha\beta} \, \left(\Lambda^{-1}\right)^\beta{}_\nu \\ &= \left(\Lambda^{-T} \, \eta \, \Lambda^{-1}\right)_{\mu\nu} \\ &= \eta_{\mu\nu} . \end{aligned}$$

The last step follows from the definition of Lorentz matrices as $SO(1,3)$ matrices:

$$\Lambda^T \, \eta \, \Lambda = \eta .$$

, and multiplying from the left by Λ^{-T} and from the right Λ^{-1}, we get,

$$\eta = \Lambda^{-T} \, \eta \, \Lambda^{-1} \tag{11}$$