

# HW 8

**DUE Friday, November 3, 3pm**

**Section 4.3:** (29), 30, 32

**Section 4.4:** 4, 8, (9), 12, 16, (18), 22, (23)

**Section 4.5:** 6, 8, 12, 14, 20, (25), (26), (32)

## Section 4.3

**Exercises: (29), 30, 32**

**4.3.29.** (recommended)

Let  $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  be a set of  $k$  vectors in  $\mathbb{R}^n$ , with  $k < n$ . Use a theorem from Section 1.4 to explain why  $S$  cannot be a basis for  $\mathbb{R}^n$ .

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**4.3.30.** Let  $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  be a set of  $k$  vectors in  $\mathbb{R}^n$ , with  $k > n$ . Use a theorem from Chapter 1 to explain why  $S$  cannot be a basis for  $\mathbb{R}^n$ .

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**Remark:** Exercises 4.3.29 and 4.3.30 show that every basis for  $\mathbb{R}^n$  must contain exactly  $n$  vectors.

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**4.3.32.** Suppose  $T$  is a one-to-one transformation. (Recall this means that, for all  $\mathbf{u}$  and  $\mathbf{v}$ , we have  $T(\mathbf{u}) = T(\mathbf{v})$  implies  $\mathbf{u} = \mathbf{v}$ .) Show that if the set  $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ , of images is linearly dependent, then  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  linearly dependent. (This fact shows that a one-to-one linear transformation maps a linearly independent set onto a linearly independent set, because in this case the set of images cannot be linearly dependent).

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## Section 4.4

**Exercises: 4, 8, (9), 12, 16, (18), 22, (23)**

**4.4.4.** Find the vector  $\mathbf{x}$  determined by the coordinate vector  $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix}$  and the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\}.$$

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**4.4.8.** Find the coordinate vector  $[\mathbf{x}]_{\mathcal{B}}$  of  $\mathbf{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$  relative to the basis  $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$ .

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4.4.9. (recommended)

Find the change-of-coordinates matrix from  $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix} \right\}$  to the standard basis in  $\mathbb{R}^n$ .

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4.4.12. Use an inverse matrix to find  $[\mathbf{x}]_{\mathcal{B}}$  for  $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$  and  $\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}$ .

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4.4.16. Mark each statement True or False. Justify each answer. Unless stated otherwise,  $\mathcal{B}$  is a basis for a vector space  $V$ .

a. If  $\mathcal{B}$  is the standard basis for  $\mathbb{R}^n$ , then the  $\mathcal{B}$ -coordinate vector of an  $\mathbf{x}$  in  $\mathbb{R}^n$  is  $\mathbf{x}$  itself.

b. The correspondence  $[\mathbf{x}]_{\mathcal{B}} \mapsto \mathbf{x}$  is called the coordinate mapping.

c. In some cases, a plane in  $\mathbb{R}^3$  is isomorphic to  $\mathbb{R}^2$ .

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4.4.18. (recommended)

Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  be a basis for a vector space  $V$ . Explain why the  $\mathcal{B}$ -coordinate vectors of  $\mathbf{b}_1, \dots, \mathbf{b}_n$  are the columns  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of the  $n \times n$  identity matrix.

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4.4.22. Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  be a basis for  $\mathbb{R}^n$ . Describe an  $n \times n$  matrix  $A$  that implements the coordinate mapping  $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ .

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4.4.23. (recommended)

Show that the coordinate mapping is one-to-one. [Hint: Suppose  $[\mathbf{u}]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}}$  for some  $\mathbf{u}$  and  $\mathbf{w}$  in  $V$  and show that  $\mathbf{u} = \mathbf{w}$ .]

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## Section 4.5

Exercises: 6, 8, 12, 14, 20, (25), (26), (32)

4.5.6. Let  $V = \left\{ \begin{bmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$ . Find the following:

a. a basis for  $V$ ;

b. the dimension of  $V$ .

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4.5.8. Let  $V = \{(a, b, c, d) : a - 3b - c = 0\}$ . Find the following:

a. a basis for  $V$ ;

b. the dimension of  $V$ .

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**4.5.12.** Find the dimension of the subspace spanned by the vectors,  $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}$ ,  $\begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$ .

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**4.5.14.** Determine the dimensions of  $\text{Nul } A$  and  $\text{Col } A$  for the matrix

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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**4.5.20.** Let  $V$  be a vector space. Mark each statement True or False. Justify each answer.

- a.  $\mathbb{R}^2$  is a two-dimensional subspace of  $\mathbb{R}^3$ .
  - b. The number of variables in the equation  $A\mathbf{x} = \mathbf{0}$  equals the dimension of  $\text{Nul } A$ .
  - c. A vector space is infinite-dimensional if it is spanned by an infinite set.
  - d. If  $\dim V = n$  and if  $S$  spans  $V$ , then  $S$  is a basis of  $V$ .
  - e. The only three-dimensional subspace of  $\mathbb{R}^3$  is  $\mathbb{R}^3$  itself.
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**4.5.25.** (recommended)

Let  $S$  be a subset of an  $n$ -dimensional vector space  $V$ , and suppose  $S$  contains fewer than  $n$  vectors. Explain why  $S$  cannot span  $V$ .

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**4.5.26.** (recommended)

Let  $H$  be an  $n$ -dimensional subspace of an  $n$ -dimensional vector space  $V$ . Show that  $H = V$ .

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**4.5.32.** (recommended)

Let  $H$  be a nonzero subspace of  $V$ , and suppose  $T$  is a one-to-one linear transformation of  $V$  into  $W$ . Prove that  $\dim T(H) = \dim H$ . (If  $T$  happens to be a one-to-one mapping of  $V$  onto  $W$ , then  $\dim V = \dim W$ . Isomorphic finite-dimensional vector spaces have the same dimension.)