HW 9

DUE: Friday, November 10, 3pm

Section 4.6

Exercises: 2, 6, 8, 10, 12, 14, 16, 18, (23), 24, (25), 26, (27)

4.6.2. Assume that the matrix A is row equivalent to B. Without calculations, list $\text{rank}\, A$ and $\dim \text{Nul}\, A$. Then find bases for $\text{Col}\, A$, $\text{Row}\, A$, and $\text{Nul}\, A$.

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.6.6. If a 6×3 matrix A has rank 3, find $\dim \text{Nul}\, A$, $\dim \text{Row}\, A$, and $\text{rank}\, A^T$.

4.6.8. Suppose a 5×6 matrix A has four pivot columns. What is $\dim \text{Nul}\, A$? Is $\text{Col}\, A = \mathbb{R}^4$? Why or why not?

4.6.10. If the null space of a 7×6 matrix A is 5-dimensional, what is the dimension of the column space of A?

4.6.12. If the null space of a 5×6 matrix A is 4-dimensional, what is the dimension of the row space of A?

4.6.14. If A is a 4×3 matrix, what is the largest possible dimension of the row space of A? If A is a 3×4 matrix, what is the largest possible dimension of the row space of A? Explain.

4.6.16. If A is a 6×4 matrix, what is the largest possible dimension of $\text{Nul}\, A$?

4.6.18. Let A be an $m \times n$ matrix. Mark each statement True or False. Justify each answer.

a. If B is any echelon form of A, then the pivot columns of B form a basis for the column space of A.

b. Row operations preserve the linear dependence relations among the rows of A.

c. The dimension of the null space of A is the number of columns of A that are not pivot columns.

d. The row space of A^T is the same as the column space of A.

4.6.23. (recommended)
A homogeneous system of twelve linear equations in eight unknowns has two fixed solutions that are not

multiples of each other, and all other solutions are linear combinations of these two solutions. Can the set of all solutions be described with fewer than twelve homogeneous linear equations? If so, how many? Discuss.

4.6.24. Is it possible for a nonhomogeneous system of seven equations in six unknowns to have a unique solution for some right-hand side of constants? Is it possible for such a system to have a unique solution for every right-hand side? Explain.

4.6.25. (recommended)
A scientist solves a nonhomogeneous system of ten linear equations in twelve unknowns and finds that three of the unknowns are free variables. Can the scientist be certain that, if the right sides of the equations are changed, the new nonhomogeneous system will have a solution? Discuss.

4.6.26. In statistical theory, a common requirement is that a matrix be of *full rank*. That is, the rank should be as large as possible. Explain why an $m \times n$ matrix with more rows than columns has full rank if and only if its columns are linearly independent.

4.6.27. (recommended)
Let A be an $m \times n$ matrix. Which of the subspaces $\text{Row}\,A$, $\text{Col}\,A$, $\text{Nul}\,A$, $\text{Row}\,A^T$, $\text{Col}\,A^T$, $\text{Nul}\,A^T$ are in $\mathbb{R}^m$ and which are in $\mathbb{R}^n$? How many distinct subspaces are in this list?

Section 4.7

Exercises: 2, 4, 6, 8, 12, 20a

4.7.2. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose $\mathbf{b}_1 = -\mathbf{c}_1 + 4\mathbf{c}_2$ and $\mathbf{b}_2 = 5\mathbf{c}_1 - 3\mathbf{c}_2$.

a. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

b. Find $[\mathbf{x}]_\mathcal{C}$ for $\mathbf{x} = 5\mathbf{b}_1 + 3\mathbf{b}_2$.

4.7.4. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ be bases for V, and let $P = [[\mathbf{d}_1]_\mathcal{A}\ [\mathbf{d}_2]_\mathcal{A}\ [\mathbf{d}_3]_\mathcal{A}]$. Which of the following equations is satisfied by P for all $\mathbf{x}$ in V?

(i) $[\mathbf{x}]_\mathcal{A} = P[\mathbf{x}]_\mathcal{D}$

(ii) $[\mathbf{x}]_\mathcal{D} = P[\mathbf{x}]_\mathcal{A}$

4.7.6. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be bases for a vector space V, and suppose $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3$, $\mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$ and $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{d}_3$.

a. Find the change-of-coordinates matrix from $\mathcal{F}$ to $\mathcal{D}$.

b. Find $[\mathbf{x}]_\mathcal{D}$ for $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$.

4.7.8. Let $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be bases for $\mathbb{R}^2$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and the change-of-coordinates $\mathcal{C}$ to $\mathcal{B}$.

4.7.12. Suppose $\mathcal{B}$ and $\mathcal{C}$ are bases for a vector space V. Mark each statement True or False. Justify each answer.

a. The columns of $P_{\mathcal{C} \leftarrow \mathcal{B}}$ are linearly independent.

b. If $V = \mathbb{R}^2$, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, then row reduction of $[\mathbf{c}_1 \ \mathbf{c}_2 \ \mathbf{b}_1 \ \mathbf{b}_2]$ to $[I \ P]$ produces a matrix P that satisfies $[\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.

4.7.20a. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ be bases for a 2-dimensional vector space.

Write an equation that relates the matrices $P_{\mathcal{C} \leftarrow \mathcal{B}}$, $P_{\mathcal{D} \leftarrow \mathcal{C}}$, and $P_{\mathcal{D} \leftarrow \mathcal{B}}$. Justify your result.