# Review of Vector Spaces and Matrix Algebra

## Review of Linear Algebra

**Definition 1.** A vector space $\boldsymbol{V}$ is a collection of objects, referred to as vectors, together with an operation of *vector addition* (which allows us to add two vectors together) and a scalar multiplication (which allows us to multiply a scalar times a vector).

Here are two examples:

**Example 1.** For example, $\boldsymbol{V} = \mathbb{R}^n = \{\boldsymbol{x} = (x_1, x_2, \cdots, x_n) : x_j \in \mathbb{R} \quad \text{for } j = 1, \cdots, n\}$. Here we define the vector addition and scalar multiplication as follows.

1. Given vectors $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$ we define $\boldsymbol{x} + \boldsymbol{y}$ by

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n).$$

2. Given a scalar $\alpha$ and a vector $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ we define $\alpha\boldsymbol{x}$ by

$$\alpha\boldsymbol{x} = (\alpha x_1, \alpha x_2, \cdots, \alpha x_n).$$

**Example 2.** The set of continuous functions on an interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ which we denote by $C[a, b]$. We have learned in calculus that the sum of two continuos functions is a continuous function and a constant times a continuous function is a continuous function.

In order for a collection of vectors $\boldsymbol{V}$ with an addition and scalar multiplication to be a vector space the following Axioms must be satisfied:

**Vector Addition** For every $\boldsymbol{x} = (x_1, x_2, \cdots, x_n), \boldsymbol{y} = (y_1, y_2, \cdots, y_n), \boldsymbol{z} = (z_1, z_2, \cdots, z_n) \in \boldsymbol{V}$

1. (closure property) $\boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{V}$.
2. (commutative property) $(\boldsymbol{x} + \boldsymbol{y}) = (\boldsymbol{y} + \boldsymbol{x})$.
3. (associative law) $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$
4. (zero vector) There is a unique zero vector $\boldsymbol{0} \in \boldsymbol{V}$ satisfying $\boldsymbol{x} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{x} = \boldsymbol{x}$.
5. (additive inverse) For every $\boldsymbol{x} \in \boldsymbol{V}$ there exists a vector $-\boldsymbol{x} \in \boldsymbol{V}$ satisfying $\boldsymbol{x} + (-\boldsymbol{x}) = (-\boldsymbol{x}) + \boldsymbol{x} = \boldsymbol{0}$.

**Scalar Multiplication** For every $\boldsymbol{x} = (x_1, x_2, \cdots, x_n), \boldsymbol{y} = (y_1, y_2, \cdots, y_n) \in \boldsymbol{V}$ and scalars $k$, $k_1$ and $k_2$

1. (closure property) $k\boldsymbol{x} \in \boldsymbol{V}$.
2. (distributive law 1) $k(\boldsymbol{x} + \boldsymbol{y}) = (k\boldsymbol{y} + k\boldsymbol{x})$.
3. (distributive law 2) $(k_1 + k_2)\boldsymbol{x} = k_1\boldsymbol{x} + k_2\boldsymbol{x}$.
4. (distributive law 3) $k_1(k_2\boldsymbol{x}) = (k_1 k_2)\boldsymbol{x}$.

5. (multiplicative identity) $1\boldsymbol{x} = \boldsymbol{x}$.

**Definition 2.** 1. A subset $\boldsymbol{W}$ of a vector space $\boldsymbol{V}$ (denoted $\boldsymbol{W} \subset \boldsymbol{V}$) is called a *subspace* if it is closed under vector addition and scalar multiplication.

2. A collection of vectors $\{\boldsymbol{x}_j\}_{j=1}^n$ in a vector space $\boldsymbol{V}$ is said to be *Linearly Independent* if the only constants $\{k_j\}_{j=1}^n$ satisfying $k_1\boldsymbol{x}_1 + k_2\boldsymbol{x}_2 + \cdots + k_n\boldsymbol{x}_n = \boldsymbol{0}$ are $k_1 = k_2 = \cdots = k_n = 0$. If a set of vectors is not linearly independent then we say it is *Linearly Dependent.*

3. A collection of linearly independent vectors $\{\boldsymbol{x}_j\}_{j=1}^n$ in a vector space $\boldsymbol{V}$ is said to be *Basis* for the vector space if every $\boldsymbol{x} \in \boldsymbol{V}$ can be written as a linear combination of the vectors $\{\boldsymbol{x}_j\}_{j=1}^n$, i.e., given $\boldsymbol{x} \in \boldsymbol{V}$ there are constants $\{c_j\}_{j=1}^n$ so that $\boldsymbol{x} = \sum_{j=1}^n c_j\boldsymbol{x}_j$.

4. The number of vectors in a basis is called the *Dimension* of the vector space.

**Example 3.** A basis (called the standard basis) for $\mathbb{R}^n$ is

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \ \boldsymbol{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Therefore the dimension of $\mathbb{R}^n$ is $n$.

**Example 4.** The vector space $C[a, b]$ is infinite dimensional. On the other hand the vector space of polynomials of degree less than $n$, denoted by $P_n$, has dimension $n$.
The standard basis for $P_n$ is $\{1, x, x^2, \cdots, x^{n-1}\}$. In particular a polynomial of degree less than $n$ has the form

$$p(x) = a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n.$$

**Example 5.** Sometimes determinants can be a useful way to test for linear independence. For example, if you have $n$ vectors from $\mathbb{R}^n$ you can form the $n \times n$ matrix with these vectors as the columns. The vectors are linearly independent if and only if the determinant is not zero. For example, consider the vectors

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

We compute the determinant of the matrix with these vectors as the columns. We find

$$\begin{vmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

so the vectors are linearly dependent. On the other hand for the vectors

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

We compute the determinant of the matrix with these vectors as the columns. We find

$$\begin{vmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 2$$

so these vectors are linearly independent.

Determinants can also be used to determine whether functions in $C^{(n-1)}(a,b)$ are linearly independent using the Wronskian. Let $f_1(x), f_2(x), \cdots, f_n(x)$ be in $C^{(n-1)}(a,b)$. Define the Wronskian $W(f_1, \cdots, f_n)$ by

$$W(f_1, \cdots, f_n)(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

If there is a point $a \leq x_0 \leq b$ such that $W(f_1, \cdots, f_n)(x_0) \neq 0$ then the functions are linearly independent.

**Remark 1.** Given a collection of linearly independent vectors $\{\boldsymbol{x}_j\}_{j=1}^{\ell}$ in a vector space $\boldsymbol{V}$ we define the *Span*, $\boldsymbol{S}$, of $\{\boldsymbol{x}_j\}_{j=1}^{\ell}$ to be the collection of all linear combinations of the vectors, i.e.,

$$\boldsymbol{S} = \left\{ \sum_{j=1}^{\ell} c_j \boldsymbol{x}_j \; : \; \text{for all scalars } c_j, \quad j = 1, \cdots, n \right\}.$$

The span of a set of vectors is a subspace and we can say that a basis is a linearly independent spanning set.

Another important property of the vector space $\mathbb{R}^n$ is that we can do geometry by introducing the so-called dot-product (or inner product). For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ we define $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{j=1}^{n} x_j y_j$. We will also often use the notation $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ for the inner product.

The inner product allows us to consider tow very important things: the length of a vector and the angle between two vectors. In particular we define the *length of a vector* by

$$|\boldsymbol{x}| = \langle \boldsymbol{x}, \boldsymbol{x} \rangle^{1/2} = \left( \sum_{j=1}^{n} x_j^2 \right)^{1/2}.$$

Notice this is exactly the formula given by the distance formula in calculus.

Then given two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ we can consider the angle $\theta$ between the vectors and using the law of cosines we obtain a formula for $\cos(\theta)$ as

$$\cos(\theta) = \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{|\boldsymbol{x}| \, |\boldsymbol{y}|}.$$

# Changing the basis in a Vector Space

We have already seen the standard basis consisting of the standard unit vectors $\{e_j\}$ in $\mathbb{R}^n$. For example any vector $\boldsymbol{x} \in \mathbb{R}^3$ can be written as

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_1 e_1 + x_2 e_2 + x_3 e_3.$$

The numbers $x_j$ are called the coordinates of the vector $\boldsymbol{x}$ with respect to the standard basis. If we have some other basis $V = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for $\mathbb{R}^3$ then the vector $\boldsymbol{x}$ above can be written as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$, i.e.,

$$\boldsymbol{x} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3$$

In this case we call $\begin{pmatrix} \alpha_1, & \alpha_2, & \alpha_3 \end{pmatrix}_V$ the coordinates of $\boldsymbol{x}$ with respect to the basis $V$.

This idea of different coordinates for different basis can seem a bit confusing and to make matters worse we often want to change back and forth between different bases. If we are given a vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ without being given a basis then we assume it is the standard basis.

For example let us take a vector $\boldsymbol{x} \in \mathbb{R}^3$ and a basis $\mathcal{B}$ for $\mathbb{R}^3$

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathcal{B} = \left\{ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Then we can write

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (3) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}_{\mathcal{B}}.$$

So the coordinates of $\boldsymbol{x}$ with respect to $\mathcal{B}$ are $\begin{pmatrix} -1, & -1, & 3 \end{pmatrix}_{\mathcal{B}}$

The general procedure for changing bases in $\mathbb{R}^n$ is to find a so-called *transition matrix*. The idea goes like this. Suppose we have two bases

$$\mathcal{B}_1 = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_n\} \quad \text{and} \quad \mathcal{B}_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}.$$

When we write this we mean that the vectors $\boldsymbol{w}_j$ and $\boldsymbol{v}_j$ are vectors in $\mathbb{R}^n$ and the entries in these vectors are with respect to the standard basis. We build the two $n \times n$ matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ whose columns are the vectors in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively.

$$\mathcal{A}_1 = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{bmatrix}, \ \mathcal{A}_2 = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}.$$

Let $\boldsymbol{x}$ be a vector in $\mathbb{R}^n$ and denote the vector with coordinates corresponding to the basis $\mathcal{B}_j$ by $\boldsymbol{x}_{\mathcal{B}_j}$. Then we have

$$\mathcal{A}_1 \boldsymbol{x}_{\mathcal{B}_1} = \boldsymbol{x} = \mathcal{A}_2 \boldsymbol{x}_{\mathcal{B}_2}$$

This implies

$$\boldsymbol{x}_{\mathcal{B}_1} = \boldsymbol{x} = \mathcal{A}_1^{-1}\mathcal{A}_2\boldsymbol{x}_{\mathcal{B}_2}$$

So we a formula for the transition matrix for $\mathcal{B}_2$ to $\mathcal{B}_1$

$$\boldsymbol{x}_{\mathcal{B}_1} = \mathcal{S}_{21}\boldsymbol{x}_{\mathcal{B}_2}, \qquad \mathcal{S}_{21} = \mathcal{A}_1^{-1}\mathcal{A}_2.$$

Similarly we have the transition matrix for $\mathcal{B}_1$ to $\mathcal{B}_2$

$$\boldsymbol{x}_{\mathcal{B}_2} = \mathcal{S}_{12}\boldsymbol{x}_{\mathcal{B}_1}, \quad \mathcal{S}_{12} = \mathcal{A}_2^{-1}\mathcal{A}_2.$$

**Example 6.** Consider the bases for $\mathbb{R}^3$

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \right\}, \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

To find the transition matrices from $\mathcal{B}_1$ to $\mathcal{B}_2$ and from $\mathcal{B}_2$ to $\mathcal{B}_1$ we first set

$$\mathcal{A}_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

In order to compute the transition matrix $\mathcal{S}_{12}$ from $\mathcal{B}_1$ to $\mathcal{B}_2$ we need to compute the inverse of $\mathcal{A}_2$. We get

$$\mathcal{A}_2^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & -1 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

and we have

$$\mathcal{S}_{12} = \mathcal{A}_2^{-1}\mathcal{A}_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & -1 \\ 1/2 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -1 & 0 \\ 2 & 1 & 1/2 \end{bmatrix}.$$

We also have $\mathcal{S}_{21} = \mathcal{S}_{12}^{-1}$ from $\mathcal{B}_2$ to $\mathcal{B}_1$ given by

$$\mathcal{S}_{21} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix}.$$

As an example let us take a vector represented in the basis $\mathcal{B}_1$ and find its coordinates with respect to $\mathcal{B}_1$ a:

$$\boldsymbol{x}_{\mathcal{B}_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{sB_1} \quad \Rightarrow \boldsymbol{x}_{\mathcal{B}_2} = \mathcal{S}_{12}\boldsymbol{x}_{\mathcal{B}_1} = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -1 & 0 \\ 2 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{sB_1} = \begin{bmatrix} -1 \\ -4 \\ 11/2 \end{bmatrix}_{sB_2}$$

To check that this is correct we can reduce both vectors to the coordinates with respect to the standard basis.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{sB_1} = (1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (2) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (3) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ -4 \\ 11/2 \end{bmatrix}_{sB_2} = (-1) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + (-4) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + (11/2) \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$$

# Row and Column Space and Rank of a Matrix

Using the definition of matrix multiplication and equality of matrices we can writea system of equations in the form:

$$\boldsymbol{AX} = \boldsymbol{B}, \qquad \text{where} \;\; \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

In order to exploit this formulation of a linear system we will introduce several important tools that can be of use in practice and in a theoretical study.

**Definition 3 (Rank of a Matrix).** The rank of an $m \times n$ matrix $\boldsymbol{A}$ is the number of linearly independent row vectors in $\boldsymbol{A}$.

**Definition 4 (Nullity of a Matrix).** The nullity of an $m \times n$ matrix $\boldsymbol{A}$ is the dimension of the null space $N(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^n \,:\, \boldsymbol{Ax} = 0\}$.

**Definition 5 (Row and Column Space).**

1. The rows of $\boldsymbol{A}$ which we denote by $\{\boldsymbol{A}_i\}_{i=1}^m$ span a subspace of $\mathbb{R}^n$ called the *Row Space* of $\boldsymbol{A}$ denoted by $R_A$.
2. The columns of $\boldsymbol{A}$ which we denote by $\{\boldsymbol{A}^j\}_{j=1}^n$ span a subspace of $\mathbb{R}^m$ called the *Column Space* of $\boldsymbol{A}$ denoted by $C_A$.

We have the following result which is useful for finding the rank of a matrix.

If $\boldsymbol{B}$ is a row-echelon form of $\boldsymbol{A}$, then

1. {the row space of $\boldsymbol{A}$ } = {the row space of $\boldsymbol{B}$ } .
2. The nonzero rows of $\boldsymbol{B}$ form a basis for $R_A$.
3. Rank($\boldsymbol{A}$) = the number of nonzero rows of $\boldsymbol{B}$.

The following are all the same

1. The rank of $\boldsymbol{A}$, i.e. number of linearly independent rows of $\boldsymbol{A}$.
2. The dimension of $R_A$, i.e. number of elements in a basis for $R_A$.
3. The number of linearly independent columns of $\boldsymbol{A}$.
4. The dimension of $C_A$.

A linear system $\boldsymbol{AX} = \boldsymbol{B}$ is consistent if and only if the rank of $\boldsymbol{A}$ is the same as the rank of the augmented matrix $(\boldsymbol{A}|\boldsymbol{B})$.

If a system is consistent and has infinitely many solutions then the solution will contain a number of arbitrary parameters. In particular for an $m \times n$ system if the rank of $\boldsymbol{A}$ is $r$ then the number of free parameters is $n - r$.

1. For an $m \times n$ matrix $\boldsymbol{A}$ the system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent for every $\boldsymbol{b}$ if and only if $C_A = \mathbb{R}^m$ and the system has at most one solution for every $\boldsymbol{b}$ if and only if the columns of $\boldsymbol{A}$ are linearly independent.

2. An $n \times n$ matrix is invertible (nonsingular) if and only if the columns of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$.

3. For an $m \times n$ matrix $\boldsymbol{A}$ we have

$$\text{Rank}(\boldsymbol{A}) + \dim(N(\boldsymbol{A})) = n,$$

   i.e., the sum of the rank and the nullity is always $n$.

4. $\dim(C_A) = \dim(R_A)$.

## How to Find Bases for Row, Column & Null Space of a Matrix

I. **Null Space of $\boldsymbol{A}$** To find a basis for the $N(\boldsymbol{A})$ just solve $\boldsymbol{A}\boldsymbol{x} = 0$ as usual. The null space is never empty because $\mathbf{0} \in N(\boldsymbol{A})$. To solve this problem you write the augmented matrix $[\boldsymbol{A}|\mathbf{0}]$ and put this matrix in row echelon form. The number of free variables is the dimension of the $N(\boldsymbol{A})$. See the example in Section 3.2.

If for example you solve the system of equations and find the solution to be

$$\boldsymbol{x} = \begin{bmatrix} \alpha - \beta \\ -2\alpha + 3\beta \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \alpha + \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \beta.$$

Then a basis for the two dimensional null space is

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

II. **Column and Row Space of $\boldsymbol{A}$** To find a basis for the column or row space you need to find the row echelon form $\boldsymbol{U}$ of $\boldsymbol{A}$.

(a) **Basis for Column Space** Find the columns in $\boldsymbol{U}$ containing the pivots. These same columns of $\boldsymbol{A}$ form a basis of the column space of $\boldsymbol{A}$.

(b) **Basis for Row Space** The nonzero rows of $\boldsymbol{U}$ form a basis of the row space of $\boldsymbol{A}$.