Introduction to method of Lagrange Multipliers

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Preliminaries: Introduction to COP

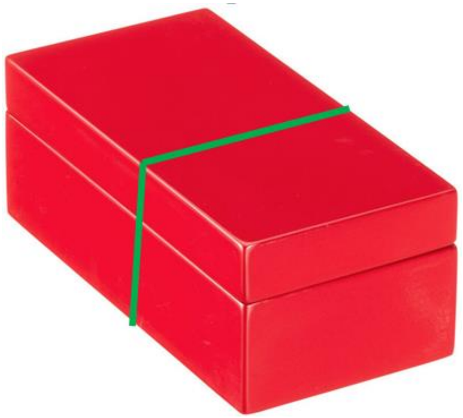
- The constrained optimization problem (COP) can be expressed in its general form as follows

$$
\begin{aligned}
&\min_{\mathbf{x}} \quad f(\mathbf{x}) \\
&\text{subject to} \\
&\qquad g_i(\mathbf{x}) \le 0 \qquad i = 1 \ldots m \\
&\qquad h_i(\mathbf{x}) = 0 \qquad i = 1 \ldots p
\end{aligned}
$$

- Example

Constrained Optimization: Example

- You are given a string of length L. You are to tie it around a rectangular gift box (l x w x h) with a constant or fixed width ‘w’ using any length of this string (up to L).
- Can you find the dimensions (length and height only since the width is fixed) of the box with the largest volume that you can tie with this string?
- Mathematically
  - Optimize the objective function
    - Max $V = wlh$
  - Subject to constraints
    - $2(l+h) \leq L$



Lagrangian Formulation

- Lagrange proposed a method for the solution of COP

$$\min_{\mathbf{x}} \quad f(\mathbf{x})$$
$$\text{subject to}$$
$$g_i(\mathbf{x}) \le 0 \qquad i = 1 \ldots m$$
$$h_i(\mathbf{x}) = 0 \qquad i = 1 \ldots p$$

  - f(x) and g_i(x) are convex functions
  - h_i(x) are affine functions

Lagrangian Formulation…

- The solution proposed by Lagrange is based on the following unconstrained minimization

$$\min_{\mathbf{x}} \qquad f\left(\mathbf{x}\right)+\sum_{i=1}^{m} L_i^g\left(\mathbf{x}\right)+\sum_{i=1}^{p} L_i^h\left(\mathbf{x}\right)$$

- Where $L_i^g\left(\mathbf{x}\right)$ and $L_i^h\left(\mathbf{x}\right)$ have the following properties

$$L_i^g\left(\mathbf{x}\right)=\begin{cases}\infty & g_i\left(\mathbf{x}\right)>0 \\ 0 & else\end{cases}, \qquad i=1...m$$

$$L_i^h\left(\mathbf{x}\right)=\begin{cases}\infty & h_i\left(\mathbf{x}\right)\neq 0 \\ 0 & else\end{cases}, \qquad i=1...p$$

  - Large penalties added when the constraints are not satisfied
    - Unconstrained optimization now leads to satisfaction of the constrains and then optimization of the original objective function

Lagrangian Formulation…

- One possible way of achieving the above mentioned properties for the two penalty functions is as follows

$$L_i^g(\mathbf{x}) = \max_{\alpha_i \geq 0} \quad \alpha_i g_i(\mathbf{x})$$

When the constraint is violated ($g_i(x)>0$) the maximization with respect to α_i leads to infinity as long as α_i is non-negative

When the constraint is not violated the maximization with respect to α_i leads to zero as long as α_i is non-negative

Similarly

$$L_i^h(\mathbf{x}) = \max_{\beta_i} \quad \beta_i h_i(\mathbf{x})$$

Lagrangian Formulation…

- Thus we can write the optimization problem as

$$\min_{\mathbf{x}} \quad f(\mathbf{x}) + \sum_{i=1}^{m} \max_{\alpha_i \geq 0} \alpha_i g_i(\mathbf{x}) + \sum_{i=1}^{p} \max_{\beta_i} \beta_i h_i(\mathbf{x})$$

$$\equiv \min_{\mathbf{x}} \quad f(\mathbf{x}) + \max_{\alpha_i \geq 0, \beta_i} \left(\sum_{i=1}^{m} \alpha_i g_i(\mathbf{x}) + \sum_{i=1}^{p} \beta_i h_i(\mathbf{x}) \right)$$

$$\equiv \min_{\mathbf{x}} \max_{\alpha_i \geq 0, \beta_i} \quad f(\mathbf{x}) + \left(\sum_{i=1}^{m} \alpha_i g_i(\mathbf{x}) + \sum_{i=1}^{p} \beta_i h_i(\mathbf{x}) \right)$$

- α_i and β_i are called Lagrange multipliers (or dual variables) and the function (below) is called the Lagrange Function

$$L(\mathbf{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = f(\mathbf{x}) + \left(\sum_{i=1}^{m} \alpha_i g_i(\mathbf{x}) + \sum_{i=1}^{p} \beta_i h_i(\mathbf{x}) \right)$$

Lagrangian function: Gift example

- The problem can be rewritten as

  - Min $f(x) = -x_1x_2$
  - Subject to constraints

    $2(x_1+x_2) \le L$, OR

    $g(x) = 2(x_1+x_2) - L \le 0$

- This implies

$$L(\mathbf{x},\boldsymbol{\alpha}) = f(\mathbf{x}) + \alpha_1 g_1(\mathbf{x})$$
$$= -x_1x_2 + \alpha_1\left(2(x_1+x_2) - L\right)$$

Lagrangian function: example...

- This can be solved as

$$\min_{\mathbf{x}} \max_{\alpha_i \geq 0} \qquad L(x_1, x_2, \alpha) = -x_1 x_2 + \alpha_1 \left(2(x_1 + x_2) - L\right)$$

$$\frac{\partial L}{\partial \alpha} = 2(x_1 + x_2) - L = 0 \qquad \Rightarrow \qquad x_1 + x_2 = L/2$$

$$\frac{\partial L}{\partial x_1} = -x_1 + 2\alpha_1 = 0 \qquad \Rightarrow \qquad x_1 = 2\alpha_1$$

$$\frac{\partial L}{\partial x_2} = -x_2 + 2\alpha_1 = 0 \qquad \Rightarrow \qquad x_2 = 2\alpha_1$$

$$\Rightarrow \qquad x_1 = x_2 = L/4$$