Homework 1

Question 1

(Durrett exercise 1.1.2) Let $\Omega = \mathbb{R}$, $\mathcal{F}$ = all subsets so that A or A^c is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.
Solution:

First we show $\mathcal{F}$ is a σ-field. Suppose $A_1, A_2, ... \in \mathcal{F}$. Let $I = \{i : A_i$ is countable$\}$. If $I = \mathbb{Z}_+$ then $\cup_i A_i$ is a countable union of countable sets so is countable, hence it is in $\mathcal{F}$. Otherwise some positive integer $k \notin I$, meaning A_k^c is countable, and then $(\cup_i A_i)^c \subset A_k^c$ so $(\cup_i A_i)^c$ is countable as well. Again this means $\cup_i A_i \in \mathcal{F}$. Regarding complements, we have $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$ by definition of $\mathcal{F}$. Thus $\mathcal{F}$ is a σ-field.

Now we show P is a p.m. Suppose $A_1, A_2, ... \in \mathcal{F}$ are disjoint. If $I \in \mathbb{Z}_+$ then $P(\cup_i A_i) = 0 = \Sigma_i P(A_i)$. If instead there is some $k \notin I$ then A_k^c is countable, so (due to disjointness) $A_i \subset A_k^c$ for all $i \neq k$, so k is the only positive integer not in I, meaning all other A_i are countable. Then $P(\cup_i A_i) = 1 = P(A_k) = \Sigma_i P(A_i)$. Thus P is a p.m.

Question 2

(Durrett exercise 1.1.4) A σ-field $\mathcal{F}$ is said to be countably generated if there is a countable collection $\mathcal{C} \subset \mathcal{F}$ so that $\sigma(\mathcal{C}) = \mathcal{F}$. Show that the Borel algebra on $\mathbb{R}^d$ is countably generated.
Solution:

For each n, we can partition the space $\mathbf{R}^d$ by a countable collection of sets of the form

$(k_1 2^{-n}, (k_1 + 1)2^{-n}] \times \cdots \times (k_d 2^{-n}, (k_d + 1)2^{-n}]$

where $k_i \in \mathbb{Z}$. Let S_n be the collection of all sets of the form above.
Claim: $\sigma(\cup_n S_n) = \mathcal{R}^d$.

Let A be an open set in $\mathcal{R}^d$. Let $A_{n,i} \in S_n$ be a sequence of all sets in S_n that are contained in A. Let $B = \cup_{n,i} A_{n,i}$ then by definition of $A_{n,i}$, we have $\cup_{n,i} A_{n,i} \subseteq A$. We claim that all points in A are contained in some $A_{n,i}$. Suppose $a \in A$. Since A is an open set, there is a cube centered at a that is contained in A. It is clear that there is an n for which there exists a set $A_{n,i}$ that is contained in A. Therefore, $\cup_{n,i} A_{n,i} = A$, i.e. any open set in $\mathbf{R}^d$ can be expressed as a union of sequence of sets $A_{n,i} \in S_n$. Therefore, $\sigma(\cup_n S_n)$ contains all open sets in $\mathbf{R}^d$ and $\mathcal{R}^d \subset \sigma(\cup_n S_n)$.

Conversely, Let S_d be a collection of an empty set and sets of the form $(a_1, b_1] \times \cdots \times (a_d, b_d] \subset \mathbf{R}^d$ where $-\infty \leq a_i < b_i \leq \infty$. Let $A = (a_1, b_1] \times \cdots \times (a_d, b_d] \in S_d$. Consider a sequence of open sets $A_i = (a_1, b_1 + 1/2^{-i}) \times \cdots \times (a_d, b_d + 1/2^{-i}) \in \mathcal{R}^d$. Since $\mathcal{R}^d$ is a σ-algebra, $\cap_i A_i = (a_1, b_1] \times \cdots \times (a_d, b_d] = A \in \mathcal{R}^d$. We have established that $S_d \subseteq \mathcal{R}^d$. Since each set in S_n is in the form above, we have $\cup_n S_n \subset \mathcal{R}^d$. Therefore $\sigma(\cup_n S_n) \subset \mathcal{R}^d$.

Question 3

(Klenke exercise 1.1.1) Let $\mathcal{A}$ be a semiring. Show that any countable (respectively finite) union of sets in $\mathcal{A}$ can be written as a countable (respectively finite) *disjoint* union of sets in $\mathcal{A}$.

Solution:

Let $A_1, A_2, ..., A_n$ be any countable union of sets in $\mathcal{A}$. Let $B_1 = A_1, B_i = A_i \setminus \cup_{j=1}^{i-1} A_j, i = 2, ..., n$. We know that all B_i are mutually disjoint. $\mathcal{A}$ is closed under finite intersection, so $\cup_{j=1}^{i-1} A_j \in \mathcal{A}, i = 2, ..., n$. Since for any $X, Y \in \mathcal{A}$, $Y \setminus X$ is a finite union of mutually disjoint sets in $\mathcal{A}$, each B_i can be expressed as $B_i = \cup_j C_{i,j}, C_{i,j} \in \mathcal{A}$ and are mutually disjoint. $\cup_i A_i = \cup_i B_i = \cup_i \cup_j C_{i,j}$. Considering all B_i are mutually disjoint, all $C_{i,j}$ are mutually disjoint.

Question 4

(Klenke exercise 1.2.1) Let $\mathcal{A} = \{(a, b] \cap \mathbb{Q} : a, b \in \mathbb{R}, a \leq b\}$. Define $\mu : \mathcal{A} \to [0, \infty)$ by $\mu((a, b] \cap \mathbb{Q}) = b - a$. Show that $\mathcal{A}$ is a semiring and μ is a content on $\mathcal{A}$ that is lower and upper semicontinuous but is not σ-additive.

Solution:

(a) $\mathcal{A}$ is a semiring.

(i) $\emptyset = (0, 0] \in \mathcal{A}$.

(ii) For any $X = (a_1, b_1] \cap \mathbb{Q}, Y = (a_2, b_2] \cap \mathbb{Q} \in \mathcal{A}$. If $a_1 \leq a_2 < b_2 < b_1$, then $Y \setminus X = \emptyset \in \mathcal{A}$; If $a_1 \leq a_2 < b_1 \leq b_2$, then $Y \setminus X = (b_1, b_2] \cap \mathbb{Q} \in \mathcal{A}$; If $a_2 < a_1 < b_1 < b_2$, then $Y \setminus X = ((a_2, a_1] \cap \mathbb{Q} \in \mathcal{A}) \cup ((b_1, b_2] \cap \mathbb{Q} \in \mathcal{A})$; If $a_2 < a_1 < b_2 \leq b_1$, then $Y \setminus X = (a_2, a_1] \cap \mathbb{Q} \in \mathcal{A}$; In other cases, $Y \setminus X = (a_2, b_2] \cap \mathbb{Q} \in \mathcal{A}$.

(iii) For any $X = (a_1, b_1] \cap \mathbb{Q}, Y = (a_2, b_2] \cap \mathbb{Q} \in \mathcal{A}$, let $a = max(a_1, a_2), b = min(b_1, b_2)$. If $b > a$, $X \cap Y = (a, b] \cap \mathbb{Q} \in \mathcal{A}$; Otherwise, $X \cap Y = \emptyset \in \mathcal{A}$.

(b) μ is a content on $\mathcal{A}$.

Given any mutually disjoint $A_i = (a_i, b_i] \cap \mathbb{Q} \in \mathcal{A}, i = 1, ..., n$, w.l.o.g. let $a_i \leq b_i \leq a_{i+1}$. $\cup_i A_i \in \mathcal{A}$ iff $b_i = a_{i+1}, \forall i$. Thus $\Sigma_i \mu(A_i) = \Sigma_i (b_i - a_i) = b_n - a_1 = \mu((a_1, b_n] \cap \mathbb{Q}) = \mu(\cup_i A_i)$.

(c) μ is lower semicontinuous.

Given $A_1 \subseteq A_2 \subseteq ... \in \mathcal{A}$, $A_i = (a_i, b_i] \cap \mathbb{Q}, \forall i$, $A_n \uparrow A$, $A = (a, b] \cap \mathbb{Q} \in \mathcal{A}$. $A = \cup_i A_i$, $a_n \downarrow a, b_n \uparrow b$. Hence $\mu(A) = b - a = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} \mu(A_n)$.

(d) μ is upper semicontinuous.

Given $A_1 \supseteq A_2 \supseteq ... \in \mathcal{A}$, $A_i = (a_i, b_i] \cap \mathbb{Q}, \forall i$, $A_n \downarrow A$, $A = (a, b] \cap \mathbb{Q} \in \mathcal{A}$. $A = \cap_i A_i$, $a_n \uparrow a, b_n \downarrow b$. Hence $\mu(A) = b - a = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} \mu(A_n)$.

(e) μ is not σ-additive.

Since $\mathbb{Q}$ is countable, let $\{x_n\}_{n\geq 1} = (0, 1] \cap \mathbb{Q} \in \mathcal{A}$ for some countable sequence of $x_1, x_2, ...$ Define $A_n = (x_n - 1/2^{n+1}, x_n] \cap \mathbb{Q} \in \mathcal{A}, \forall n$; Define $B_1 = A_1, B_i = A_i \backslash \cup_{j=1}^{i-1} A_j, i = 2, ..., n$. Since $x_n \in A_n, \forall n$, $\cup_i A_i = (0, 1] \cap \mathbb{Q}$. $\mu(\cup_i B_i) = \mu(\cup_i A_i) = 1$. However, $\Sigma_i \mu(B_i) \leq \Sigma_i \mu(A_i) = \Sigma_i 1/2^{i+1} = 1/2$. Since $\mu(\cup_i B_i) \neq \Sigma_i \mu(B_i)$, μ is not σ-additive.