Homework 3

Question 1

(Change-of-variable formula, Klenke Theorem 4.10) Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces, let μ be a measure on $(\Omega, \mathcal{A})$ and let $X : \Omega \to \Omega'$ be measurable. Let $\mu' = \mu \circ X^{-1}$ be the image measure of μ under the map X, i.e., for every $E \in \mathcal{A}'$, we define

$\mu'(E) \triangleq \mu(X^{-1}(E))$.

Assume that $f : \Omega' \to \bar{\mathbb{R}}$ is μ'-integrable. Show that $f \circ X$ is μ-integrable and

$\int (f \circ X)\ d\mu = \int f\ d\mu'$.

(Hint: You can follow the proof of Theorem 1.6.9 in Durrett, but you need to write the answer in your own words.)

Solution:

We will prove this result by verifying it in four increasingly general special cases.

Case 1: Indicator functions. If $E \in \mathcal{A}'$ and $f = 1_E$ then

$\int (f \circ X) d\mu = \int 1_E(X(\omega)) d\mu(\omega) = \mu(X^{-1}(E)) = \mu'(E) = \int 1_E d\mu' = \int f d\mu'$.

Case 2: Simple functions. If $f(x) = \Sigma_{i=1}^n c_i 1_{E_i}$ where $c_i \in \mathbf{R}, E_i \in \mathcal{A}'$ then

$\int (f \circ X) d\mu = \int \Sigma_{i=1}^n c_i 1_{E_i}(X(\omega)) d\mu = \Sigma_{i=1}^n c_i \int 1_{E_i}(X(\omega)) d\mu = \Sigma_{i=1}^n c_i \int 1_{E_i} d\mu' = \int \Sigma_{i=1}^n c_i 1_{E_i} d\mu' = \int f d\mu'$.

Case 3: Nonnegative functions. Now if $f \geq 0$ and we let

$f_n(x) = ([2^n f(x)]/2^n) \wedge n$

where $[x]$ = the largest integer $\leq x$ and $a \wedge b = min\{a, b\}$, then the f_n are simple and $f_n \uparrow f$, so using the result for simple functions and the monotone convergence theorem:

$\int (f \circ X) d\mu = \lim_n \int (f_n \circ X) d\mu = \lim_n \int f_n d\mu' = \int f d\mu'$.

Case 4: Integrable functions. The general case now follows by writing $f(x) = f(x)^+ - f(x)^-$ where $f(x)^+$ and $f(x)^-$ are nonnegative functions. Then

$\int (f \circ X) d\mu = \int (f^+ \circ X) d\mu - \int (f^- \circ X) d\mu = \int f^+ d\mu' - \int f^- d\mu' = \int f d\mu'$.

In addition, $\int |f \circ X| d\mu = \int (f^+ \circ X) d\mu + \int (f^- \circ X) d\mu = \int f^+ d\mu' + \int f^- d\mu' = \int |f| d\mu' < \infty$.

So $f \circ X$ is μ-integrable.

Question 2

(Klenke exercise 4.2.1) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f \in L^1(\mu)$. Show that for any $\epsilon > 0$, there is an $A \in \mathcal{A}$ with $\mu(A) < \infty$ and

$|\int_A f\ d\mu - \int f\ d\mu| < \epsilon$.
Solution:
// We can answer this question without using Markov inequality, or MCT, or DCT.
// We note that this question is trivial if $(\Omega, \mathcal{A}, \mu)$ is a probability space, because in this case we can simply take $A = \Omega$.
Sketch of proof.

We can first assume that the function f is non-negative and $\int f\ d\mu$ is finite. Let c denote the value of $\int f\ d\mu$. By the definition of Lebesgue integral for nonnegative function, c is the supremum of $\int h\ d\mu$, with h ranging over all simple functions such that $0 \le h \le f$. For any $\epsilon > 0$, we can find a simple function h in the form

$h(\omega) = \sum_{i=1}^m\ a_i \mathbf{1}_{A_i}(\omega)$, so that $0 \le h \le f$, $a_i > 0$ for all i, and

$c - \epsilon \le \int h\ d\mu \le c$.

With this choice of h, we get

(*) $\quad \int f\ d\mu - \int h\ d\mu \le \epsilon$.

The measure of A_i could not be infinite, otherwise the integral of h would be infinite, violating the assumption that $\int f\ d\mu$ is finite. We take A to be the union of A_i, for $i = 1, 2, \ldots, m$. As each A_i has finite measure, the union A also has finite measure.

We next use the fact that the simple function h is less than or equal to f on the set A. By monotone property of Lebesgue integral,

$\int h\ d\mu = \int_A h\ d\mu \le \int_A f\ d\mu$.

Combining this with (*), we obtain

$0 \le \int f\ d\mu - \int_A f\ d\mu \le \epsilon$.

This proves the statement in the question for non-negative function f.

For general real-valued f, we apply the above argument for the positive and negative parts of f. Because f is assumed to be μ-integrable, $\int f^+\ d\mu$ and $\int f^-\ d\mu$ are both finite. Take A to be the union of all the sets appearing in the indicator functions that approximate the positive and negative part.

Question 3

(Minkowski's inequality, Durrett Exercise 1.5.3) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and f, g be real-valued μ-measurable functions. Suppose that $p \in (1, \infty)$. The p-norm of function f, denoted by $\|f\|_p$ is defined as $(\int |f|^p\ d\mu)^{1/p}$. The inequality $|f + g|^p \le 2^p(|f|^p + |g|^p)$ shows that if $\|f\|_p$ and $\|g\|_p$ are finite, then so is $\|f + g\|_p$. Apply Holder's inequality to $|f|\ |f + g|^{p-1}$ and $|g|\ |f + g|^{p-1}$ to show

$\|f + g\|_p \le \|f\|_p + \|g\|_p$.

(Hint: See Theorem 7.17 in Klenke)
Solution:

$$
\begin{aligned}
\|f+g\|_p^p &= \int (f+g)^p d\mu = \int f(f+g)^{p-1} d\mu + \int g(f+g)^{p-1} d\mu \\
&\leq \|f\|_p \cdot \|(f+g)^{p-1}\|_q + \|g\|_p \cdot \|(f+g)^{p-1}\|_q \\
&= (\|f\|_p + \|g\|_p) \cdot \|f+g\|_p^{p-1}.
\end{aligned} \tag{1}
$$

Note that in the last step, we used the fact that $p - p/q = 1$. Dividing both sides by $\|f+g\|_p^{p-1}$ yields the result.

Question 4

(Durrett Exercise 1.6.11) Prove that if $E[\,|X|^k\,] < \infty$, then for $0 < j < k, E[\,|X|^j\,] < \infty$, and furthermore $E[\,|X|^j\,] \leq E[\,|X|^k\,]^{j/k}$.
(The purpose of this question is to show that the j-norm of a random variable is less than or equal to the corresponding k-norm for $j < k$.)
Solution:
// The asserted inequality is false in general when the measure of the whole space is not 1
Apply Jensen's inequality with the function $\phi(x) = x^{k/j}$, which is a convex function on the nonnegative real numbers when $k \geq j$. Suppose that $E|X^k|$ is finite.

$$
\begin{aligned}
E[\,|X|^k\,] &= E[\,\phi(|X|^j)\,] \\
&\geq \phi(E[\,|X|^j\,]) \\
&= E[\,|X|^j\,]^{k/j}.
\end{aligned}
$$

This is the same as $E[\,|X|^j\,] \leq E[\,|X|^k\,]^{j/k}$.
Because $E[\,|X|^k\,]$ is finite, the right-hand side of the above inequality is finite. The left-hand side is finite as well.
// For finite measure space, we can still prove that finite L_k norm implies finite L_j norm. See this wikipedia page. This uses the Holder inequality.

Question 5

(Lebesgue integral with counting measure) Let $\mathbb{Z}$ denote the set of integers and μ be the counting measure, i.e., for subset $S \subseteq \mathbb{Z}$,

$$
\mu(S) = \begin{cases} |S| & \text{if } |S| \text{ is finite} \\ \infty & \text{otherwise.} \end{cases}
$$

For any function $f : \mathbb{Z} \to \mathbb{R}$, let S_+ be the the set $\{x \in \mathbb{Z} : \ f(x) > 0\}$ and S_- be the the set $\{x \in \mathbb{Z} : \ f(x) < 0\}$.
(We may call S_+ the positive support of f and S_- the negative support of f.)
If $f \in L^1(\mu)$, show that

$\int f\ d\mu = \sup\{\sum_{x\in A}\ f(x) :\ |A| < \infty, A \subseteq S_+\} - \sup\{-\sum_{x\in B}\ f(x) :\ |B| < \infty, B \subseteq S_-\}$.
The first supremum is taken over all finite subset $A \in \mathbb{Z}$ such that $f(x) > 0$ for all $x \in A$. Likewise, the second supremum is taken over all finite subset B such that $f(x) < 0$ for all $x \in B$.
Solution:

Consider the positive part f^+ of the function f first. By definition, the Lebesgue integral of f^+ with counting measure on $\mathbb{Z}$ is the supremum of the set

$S_1 = \{\int_{\mathbb{Z}}\ g\ d\mu :\ 0 \leq g \leq f^+, g \text{ simple}\}$.

For simple function g, $\int_{\mathbb{Z}}\ g\ d\mu$ is just a finite summation.

This question is suggesting that it suffices to consider the set

$S_2 = \{\sum_{x\in S_1}\ f(x) :\ |S_1| < \infty,\ f(x) > 0\ \forall x \in S_1\}$, which may look simpler than S_1.

Let $s_1 = \sup S_1$ and $s_2 = \sup S_2$. Because S_2 is a subset of S_1. We have $s_2 \leq s_1$. Our goal is to prove the reverse inequality. Because it is assumed that f is in $L^1(\mu)$, s_1 is a finite number.

We claim that $s_2 \geq s_1 - \epsilon$ for every arbitrarily small positive real numbers ϵ. Since $s_1 - \epsilon$ is strictly less than s_1, $s_1 - \epsilon$ is no longer an upper bound of S_1. This means that we can find a simple function $g(x)$ in the form

$g(x) = \sum_{i=1}^k\ c_i \mathbf{1}_{A_i}(x)$ for some positive constant c_i and subset A_i of $\mathbb{Z}$, such that $0 \leq g(x) \leq f^+(x)$ for all x, and $\int g(x)\ d\mu \geq s_1 - \epsilon$. Because s_1 is assumed to be finite, A_i's are all finite sets.

From g, we can construct a larger function h by taking A to be the union of $A_1, \ldots, A_k$ and let

$h(x) = f(x)\mathbf{1}_A(x)$.

Because the sample space $\mathbb{Z}$ is discrete and A is finite, the function $h(x)$ is a simple function. Thus

$\int g(x)\ d\mu \leq \int h(x) d\mu = \sum_{x\in A}\ f^+(x)$. We observe that $\sum_{x\in A}\ f^+(x)$ is an element in S_2, the supremum s_2 of S_2 must be larger than or equal to $\sum_{x\in A}\ f^+(x)$, and hence larger than or equal to $\int g(x)\ d\mu$. This proves that $s_2 \geq s_1 - \epsilon$.

Since $s_2 \geq s_1 - \epsilon$ holds for arbitrarily small positive real number ϵ, we must have $s_2 \geq s_1$. This proves that $s_2 = s_1$.

The proof for the negative part is the same as above. This completes the proof.
// There are two messages from this exercise.
// The first message is that there is no "conditional convergence" in Lebesgue measure.
// Summation like $\sum_{n=1}^{\infty}\ (-1)^n/n$ does not exist in the Lebesgue sense.
// See https://math.stackexchange.com/questions/1472173/no-conditional-convergence-in-lebesgue-integration
// and https://math.stackexchange.com/questions/1095666/lebesgue-integral-and-absolute-value
// The second one is the proof technique of "Give yourself an epsilon of room".
// This is a common trick in real analysis

// See https://terrytao.wordpress.com/2009/02/28/tricks-wiki-give-yourself-an-epsilon-of-room/

Question 6

(Improper Riemann integral) In Lecture 10 we demonstrated the relationship between Riemann integral $\int_a^b f(x)\, dx$ and Lebesgue integral $\int_{[a,b]} f(x) d\lambda(x)$. The purpose of this question is to consider the the case of improper Riemann integral.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying

(i) $\int_a^b f(x)\, dx$ is Riemann integrable for any a and b s.t. $-\infty < a < b < \infty$;

(ii) The double limit $\lim_{a\to-\infty} \lim_{b\to\infty} \int_a^b |f(x)|\, dx$ exists.

The double limit above is usually denoted as $\int_{-\infty}^{\infty} |f(x)|\, dx$.

In this question we denote the Lebesgue measure on $\mathbb{R}$ by λ.

(a) For positive integer n, define the truncated function $g_n(x) \triangleq |f(x)|\mathbf{1}_{[-n,n]}(x)$, where $\mathbf{1}_{[-n,n]}$ is the indicator function of $[-n, n]$. From the second theorem in lecture 10, we obtain

$\int_{[-n,n]} |g_n(x)|\, d\lambda(x) = \int_{-n}^{n} |f(x)|\, dx.$

The LHS is the Lebesgue integral of $|g_n(x)|$ on the integral $[-n, n]$ w.r.t. the Lebesgue measure λ, and the RHS is the Riemann integral of $|f(x)|$ from $-n$ to n.

Verify that g_n is an increasing sequence of non-negative functions converging pointwise to $|f|$. Prove that

$\int_{\mathbb{R}} |f(x)|\, d\lambda(x) = \int_{-\infty}^{\infty} |f(x)|\, dx < \infty,$

and hence show that $f(x)$ is λ-integrable .

(Hint: Use monotone convergence theorem.)

(b) For positive integer n, define $h_n(x) \triangleq f(x)\mathbf{1}_{[-n,n]}(x)$. Verify that the functions $h_n(x)$'s are dominated by $|f(x)|$ for all n, and converge to $f(x)$ as n increases. By applying dominated convergence theorem, prove that

$\int_{\mathbb{R}} f(x)\, d\lambda(x) = \int_{-\infty}^{\infty} f(x) dx.$

Solution:

(a) The function g_n is increasing because it is a product of two nonnegative functions $|f(x)|$ and $\mathbf{1}_{[n,n]}$, and $(\mathbf{1}_{[n,n]})_{n\geq 1}$ is an increasing sequence of functions. It is assumed that the Riemann integral of g_n from $-n$ to n exists for all n. Using Theorem 2 in lecture 10, we have

$\int g_n(x)\, d\lambda(x) = \int_{[n,n]} g_n(x)\, d\lambda(x) = \int_{-n}^{n} g_n(x)\, dx$. If we take limits as $n \to \infty$, the left-hand side approaches the Lebesuge integral $\int |f(x)|\, d\lambda(x)$ by MCT. The right-hand side approaches $\int_{-\infty}^{\infty} |f(x)|\, dx$.

Since $\int_{-\infty}^{\infty} |f(x)|\, dx$ is finite by assumption, $\int |f(x)|\, d\lambda$ is also finite. Therefore $f(x)$ is λ-integrable.

(b) For each n, the function $h_n(x)$ is dominated by $f(x)$ because

$|h_n(x)| \leq |f(x)| \cdot |\mathbf{1}_{[n,n]}(x)| \leq |f(x)|.$

For a given x, $h_n(x)$ is equal to $f(x)$ for all $n \geq x$, and hence converges to $f(x)$ as $n \to \infty$.

By dominated convergence theorem,

(1) $\lim_n \int h_n(x)\ d\lambda = \int \lim_n h_n(x)\ d\lambda = \int f(x)\ d\lambda$.

Since h_n has finite support, by Theorem 2 in lecture 10,

$\int h_n(x)\ d\lambda(x) = \int_{[n,n]} h_n(x)\ d\lambda(x) = \int_{-n}^{n} h_n(x)\ dx$.

So

(2) $\lim_n \int h_n(x) = \lim_n \int_{n}^{n} h_n(x)\ dx = \int_{-\infty}^{\infty} f(x)\ dx$. (The second equality comes from the definition of improper Riemann integral.)

Putting (1) and (2) together, we get

$\int f(x)\ d\lambda = \int_{-\infty}^{\infty} f(x)\ dx$.