

1 : Observe that the matrix A contains two block matrices

$$A_1 = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

These matrices can be inverted using the 2×2 inverse formula, giving

$$A_1^{-1} = -\frac{1}{2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}, \quad A_2^{-1} = -\frac{1}{2} \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$$

Now, A^{-1} should satisfy $AA^{-1} = I$. Therefore

$$\begin{bmatrix} 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 6 & 0 \end{bmatrix} \begin{bmatrix} \times & & \times & \\ & \times & \times & \\ \times & & \times & \\ & \times & \times & \end{bmatrix} = I,$$

where the \times show which entries of A^{-1} will interact with the nonzero entries of A . Notice that A^{-1} will also contain two block matrices. We just need to place the inverses of A_1 and A_2 in the correct blocks, giving

$$A^{-1} = \begin{bmatrix} -3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 5/2 & 0 & 0 & -3/2 \\ 0 & 1/2 & 0 & 0 \end{bmatrix}$$

2 : False. Any projection matrix P will satisfy $P^2 = P$. For example the matrix $\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

implements projection onto the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and satisfies the given equality.

3 : False. It is still possible to have inconsistencies, for example

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

has no solutions, even though $N(A) = \{0\}$.

4 : False. One counterexample is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

It can be seen that $x_1 = x_2 = 0$ is the only solution for $Ax = 0$. However, no combination of the columns of A can give a vector $y \in \mathbb{R}^3$ with a nonzero third entry.

5 : (i) A can be written as

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} := uu^T$$

Therefore $\text{rank}(A) = 1$, and one of the eigenvalues is $\lambda_1 = 0$. From the equation above we see that $v_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}$ gives $Av_1 = 0$.

Also if we try u as an input, we can see from the above form of A that we will get some scaled output of u back, meaning u is an eigenvector of A as well, associated with $\lambda_2 = 2$.

(ii) Note that $B = A - I$. Therefore the eigenvectors of B will be the same as the eigenvectors of A , and the eigenvalues of B can be found just by subtracting 1 from the eigenvalues of A .

(iii) We can view C as a block matrix of the form

$$C = \begin{bmatrix} A & A \\ A & A \end{bmatrix} \implies Cv = \begin{bmatrix} A & A \\ A & A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where A is the matrix from part (i). Now, if we set $x_1 = x_2 = x$ the output is

$$Cv = \begin{bmatrix} A & A \\ A & A \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} 2Ax \\ 2Ax \end{bmatrix} = 2 \begin{bmatrix} Ax \\ Ax \end{bmatrix}$$

If x is set equal to an eigenvector of A , say v_i , we will have

$$C \begin{bmatrix} v_i \\ v_i \end{bmatrix} = 2 \begin{bmatrix} Av_i \\ Av_i \end{bmatrix} = 2\lambda_i \begin{bmatrix} v_i \\ v_i \end{bmatrix}$$

This shows that for any eigenvalues of A , λ_i , we have that $2\lambda_i$ is an eigenvalue of C . In particular, the eigenvalues of C are 4 and 0 (repeated 3 times).

Now, note that similar to the case with A , C can be written as $v_C v_C^T$, where $v_C = [1 \ 1 \ 1 \ 1]^T$. Therefore, $\text{rank}(C) = 1$, and we can find three linearly independent (in fact orthogonal) eigenvectors of C corresponding to eigenvalues 0. For example

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

(iv) D can be written as

$$D = \begin{bmatrix} \alpha & \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha & \alpha \end{bmatrix} + (1 - \alpha)I$$

Therefore, the eigenvectors of D will be the same as those of C , and the eigenvalues are $4\alpha + 1 - \alpha = 3\alpha + 1$ and $1 - \alpha$ (repeated 3 times).

(v) E contains the block $E_1 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$. This matrix can be written

$$E_1 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, using the previous parts we can see that this matrix has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 1$, corresponding to eigenvectors $v_1 = [1 \ 1]^T$ and $v_2 = [1 \ -1]^T$.

Continuing, we see that

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & E_1 \end{bmatrix}$$

Therefore

$$E \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} E_1 & 0 \\ 0 & E_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} E_1 x_1 \\ E_1 x_2 \end{bmatrix}$$

Therefore we see that using either $x_1 = x_2 = v_i$, $x_1 = v_i$ and $x_2 = 0$, or $x_1 = 0$ and $x_2 = v_i$ where v_i is an eigenvector of E_1 gives the eigenvectors of E .

(vi) F can be written in block form as

$$F = \begin{bmatrix} E_1 & 2E_1 \\ 2E_1 & E_1 \end{bmatrix}$$

Therefore

$$F \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} E_1 & 2E_1 \\ 2E_1 & E_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} E_1(x_1 + 2x_2) \\ E_1(2x_1 + x_2) \end{bmatrix}$$

So, if $x_1 = x_2$, then we get

$$F \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 3E_1x_1 \\ 3E_1x_1 \end{bmatrix}$$

Therefore, two of the eigenvalues of F are 3 and 15, with associated eigenvectors given by setting $x_1 = x_2 = v_i$, where v_i is the corresponding eigenvector of E_1 .

If $x_2 = -x_1$, then

$$F \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = \begin{bmatrix} -E_1x_1 \\ E_1x_1 \end{bmatrix}$$

Therefore, the other two eigenvalues of F are -1 and -5, with associated eigenvectors $x_1 = v_i$ and $x_2 = -v_i$, where again v_i is the appropriate eigenvector of E_1 .

6 : Using the properties of the determinant, the answer can be computed as

$$\begin{aligned} \det A &= \det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a & b & c \end{bmatrix} + \det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ b & c & a \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 0 & 0 \\ a & b-a & c-b \\ b & c-a & a-b \end{bmatrix} \\ &= (b-a)(a-b) - (c-a)(c-b) \end{aligned}$$

7 : True. Let $Ax = \lambda x$. Then $A^T Ax = A^T \lambda x$, and since $A^T = A$, we have that $A^T Ax = \lambda^2 x$. This shows that λ^2 is an eigenvalue for $A^T A$.

Now, λ^2 is the square of a singular value for the matrix A . Since A is positive definite, $\lambda > 0$,

so that $\sqrt{\lambda^2} = \lambda > 0$. This shows that the singular values of A are equal to the eigenvalues of A .