HW#9—Phys402—Fall 2018 Prof. Ted Jacobson
Due Wenesday Nov. 15, 10am, at the beginning of class PSC 3151, (301)405-6020
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1. Consider scattering of a particle of mass m by a "soft cube" potential,

$$V(\vec{x}) = V_0\,\theta(a - |x|)\theta(a - |y|)\theta(a - |z|).$$

Assume that the potential is "weak", in the sense that $|V_0| \ll \hbar^2/ma^2$, so that the Born approximation is accurate at all energies.

(a) Find the differential cross section for particles with incoming momentum $\vec{p_i} = p\hat{z}$ to scatter into the $\hat{x}$ direction.

(b) If the incident beam has a number flux of I particles per unit area per unit time, how many particles per unit time are scattered into a small solid angle $\Delta\Omega$ about the $\hat{x}$ direction?

(c) Under what conditions could the scattering be approximately isotropic, and what then would be the differential cross section, and the total cross section?

2. Consider scattering of a particle of mass m and incident wavevector $\vec{k}_i = k\hat{z}$ from a potential $V(\vec{r}) = W(\vec{r}) + W(\vec{r} - b\hat{z})$, where $W(\vec{r})$ is a localized potential with no particular symmetry and b is some constant length.

(a) Find an expression for the scattering amplitude in the first Born approximation. Express the result in terms of the Fourier transform of W, without any reference to V.

(b) For which incoming wavenumbers k does the differential cross section vanish at $\theta = \pi/2$, $\phi = 0$ (regardless of the form of $W(\vec{r})$)? Explain in physical terms (i.e. without reference to equations or formulas) why that vanishing occurs.

3. GS 10.20 (*Gaussian potential scattering in the Born approximation*) Give also the scattering amplitude and the differential cross section. *Note*: The μ in the exponent is a typo. It should be deleted. *Hint*: It's nice to use the fact that the Fourier transform of a Gaussian is a Gaussian! (That holds in any dimension.) *Answer for part (a)*: $\sigma \propto (1 - e^{-2k^2a^2})/k^2$. Add parts:

(b) Evaluate the zero momentum limit of the scattering amplitude, and compare it to the geometrical quantity a^2. By what factor is the cross section smaller?

(c) Evaluate the *forward scattering amplitude*, i.e. the scattering amplitude at $\theta = 0$. How does it behave in the low energy and high energy limits?

(d) If $ka = 10$, within approximately what angle θ_m does most of the scattering take place? In general, how does this angle depend on ka?

(e) How does the total cross section depend on energy in the high energy limit $ka \gg 1$?

(f) Under what conditions on the parameters is the Born approximation a good approximation for this potential?

4. Consider s-wave scattering of a particle of mass m and wavenumber k on a spherical well potential of radius a and depth V_0. Up to an overall normalization, the wave function outside the potential has the form $\sin(kr + \delta)/r$, while that inside is $A\sin(qr)/r$, where $q^2 = k^2 + 2mV_0/\hbar^2$. The s-wave phase shift δ and interior amplitude A remain to be determined. (*continued on next page*)

(a) Show that the matching conditions at $r = a$ imply $A = (k/q)/\sqrt{\cos^2 qa + (k/q)^2 \sin^2 qa}$.

(b) Suppose that $ka \ll s$, where $s := \sqrt{2mV_0 a^2/\hbar^2}$ is the dimensionless "strength" of the potential. Then $k \ll q$, so the previous part shows that $|A| \ll 1$ unless $\cos qa$ is close to zero. That means the wave function is very small inside the potential compared to outside, so the deep potential well acts like a hard sphere, unless $\cos qa$ is close to zero.

i. Show that if $\cos qa$ is *not* close to zero then the phase shift is $\delta \approx -ka \mod \pi$, like for the hard sphere. Show that if also $ka \ll 1$ then the cross section is $\sigma \approx 4\pi a^2$.

ii. Show that if $\cos qa = 0$ then $A = 1$ and the phase shift is $\delta = -ka + \pi/2 \mod \pi$. Show that if also $ka \ll 1$ then the cross section is $\sigma \approx 4\pi/k^2$, which is $(ka)^{-2}$ times larger than in the generic case.

iii. Show that $\cos qa = 0$ is also the condition for the potential to posesses a zero energy bound state.

(c) Show that there is an s-wave bound state provided $s > \pi/2$.