PH 340: Quantum Statistical Field Theory

Problem Set # 4

Total points: 65

Due: Thu. Feb. 21 2019

1. (**10 points**) Calculate the propagator $\langle x', t' | x_0, 0 \rangle$ for a free particle of mass m moving in 1D starting from the path integral form of the propagator

$$\langle x', t' | x_0, 0 \rangle = \int [\mathcal{D}x] \exp \left[\frac{i}{\hbar} \int_0^t L\left(\{\dot{x}_j\}, \{x_j\} \right) dt \right].$$

Do this by discretizing the above expression so that

$$\langle x', t' | x_0, 0 \rangle = \int dx_{N-2} \int dx_{N-3} \cdots \int dx_1 \exp \left[\frac{i}{\hbar} \sum_{n=0}^{N-2} \frac{m}{2} \left\{ \frac{(x_{n+1} - x_n)^2}{(\Delta t)^2} \right\} dt \right],$$

with $x'(t') = x_{N-1}$ and then sequentially integrating out $x_{N-2}, x_{N-3} \ldots x_1$.

2. (**10 points**) Consider a system with a Lagrangian $L\left(\{\dot{q}_j\}, \{q_j\} \right)$ which only contains quadratic terms of its arguments. Show that the propagator for such a system is of the form

$$\langle q_j', t' | \left(q_0 \right)_j, 0 \rangle = f(t') e^{iS_c},$$

where S_c is the action of the classical path that connects the point with coordinates $\{(q_0)_j\}$ at time $t = 0$ to the one with coordinates $\{q_j'\}$ at time $t = t'$ and $f(t')$ is only a function of t'.

3. (**10 points**) Use the above result to calculate the propagator $\langle x', t' | x_0, t_0 \rangle$ for a particle of mass m moving in 1D in a harmonic potential of frequency ω. You do not need to calculate $f(t)$. Is the propagator translationally invariant (i.e. dependent only on $|x' - x|$)? Why or why not?

4. (**15 points**) Consider a (non-relativistic) particle of charge q and mass m moving under the influence of electric and magnetic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, which can be derived from the the scalar and vector potentials $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$.

   (a) (**3 points**) Write down the Hamiltonian $H(\mathbf{p}, \mathbf{r})$ for the system.

   (b) (**3 points**) From the Hamiltonian, derive the Lagrangian $L(\dot{\mathbf{r}}, \mathbf{r})$. How does the action change under a gauge transformation?

   (c) (**5 points**) Use the result of problem 2 to obtain the propagator $\langle \mathbf{r}', t' | \mathbf{r}_0, 0 \rangle$ for the above particle moving in two dimensions in a constant and uniform perpendicular magnetic field B. Work in the gauge $\mathbf{A} = \frac{1}{2} (\mathbf{B} \times \mathbf{r})$. You do not need to calculate $f(t)$.

   (d) (**4 points**) The propagator calculated above is not translationally invariant. Prove that a uniform translation $\mathbf{r} \to \mathbf{r} + \mathbf{d}$ is equivalent to a gauge transformation. Thus, from the result of (c), show that a uniform translation only changes the propagator by a constant phase factor.

5. (**10 points**) The propagator $\langle \{x\}, t | \{x'\}, 0\rangle$ of a general system can be written as a phase space path integral

$$\langle \{x\}, t | \{x'\}, 0\rangle = \int [\mathcal{D}p]\,[\mathcal{D}x] \exp\left[\frac{i}{\hbar}\int_0^t \left(\sum_j p_j \dot{x}_j - H\left(\{p_j\}, \{x_j\}\right)\right) dt\right],$$

where $H\left(\{x_j\}, \{x_j\}\right)$ is the Hamiltonian of the system. Assume that the Hamiltonian is quadratic in p_j, i.e.

$$H\left(\{p_j\}, \{x_j\}\right) = \frac{1}{2}\sum_{ij} f_{ij} p_i p_j + \sum_j g_j(x) p_j + h(x),$$

where the $f_{i,j}$'s are constants and $g_j(x)$ and $h(x)$ are functions of the coordinates x_j.

(a) (**4 points**) Obtain the Lagranian $L\left(\{\dot{x}_j\}, \{x_j\}\right)$.

(b) (**6 points**) Integrate out the momenta in the path integral to show that (up to multiplicative prefactors)

$$\langle \{x_j\}, t | \{x'_j\}, 0\rangle = \int [\mathcal{D}x] \exp\left[\frac{i}{\hbar}\int_0^t L\left(\{\dot{x}_j\}, \{x_j\}\right) dt\right].$$

6. (**10 points**) The above result along with the one from problem # 4 implies that the phase space path integral

$$\langle \mathbf{r}, t | \mathbf{r}', 0\rangle = \int [\mathcal{D}\mathbf{p}]\,[\mathcal{D}\mathbf{r}] \exp\left[\frac{i}{\hbar}\int_0^t \left(\mathbf{p}.\dot{\mathbf{r}} - H\left(\mathbf{p}, \mathbf{r}\right)\right) dt\right],$$

of a particle in a magnetic field depends on the field. Now, consider a different type of phase space integral

$$Z = \int d\mathbf{p}\; d\mathbf{r} \exp\left[-\beta H\left(\mathbf{p}, \mathbf{r}\right)\right],$$

which is the partition function at temperature $T = 1/k_B\beta$ of the particle.

(a) (**3 points**) Show that Z is independent of the magnetic field.

(b) (**3 points**) What is the physical implication of the above result?

(c) (**4 points**) To contrast the magnetic field dependence of the two different kinds of phase space integrals, consider

$$\int d\mathbf{p}\; d\mathbf{r} \exp\left[\frac{i}{\hbar}\left\{\mathbf{p}.\dot{\mathbf{r}} - H\left(\mathbf{p}, \mathbf{r}\right)\right\}\Delta t\right],$$

which is just the propagator evaluated for an infinitesimal time interval. Show that this integral is not independent of the magnetic field unlike Z. The origin of the field dependence of the full propagator lies in the field dependence of the above type of integral.