Chapter 9

Operator Quantization of Second Class Systems

9.1 Introduction

In chapters 3 and 5 we have discussed in detail the classical Poisson bracket formulation of the Hamilton equations of motion for singular systems, and their symmetries, respectively. We now turn to the problem of formulating the corresponding quantum theory. As we shall show, purely second class systems allow for a straightforward operator quantization (apart from possible ordering problems), while theories which involve also first class constraints must first be effectively converted to second class systems by imposing an appropriate number of gauge conditions. These are subsidiary conditions imposed from the "outside", i.e., they are not part of the Euler-Lagrange equations of motion. When quantizing second class systems we shall be led to introduce an *extended* Hamiltonian which includes all the constraints, primaries and secondaries, with their respective Lagrange multipliers, and - in the case of gauge fixing - the gauge conditions as well. The gauge conditions are chosen in such a way that, together with the second class constraints, they turn the theory into a pure second class system. In terms of the extended Hamiltonian the equations of motion take a form analogous to those expressed in terms of the total Hamiltonian. These will be shown to be completely equivalent to the equations of motion formulated by Dirac.

9.2 Systems with only second class constraints

Consider a purely second class system subject to the constraints $\Omega^{(2)}_{A_2} = 0$. Such a system possesses no local (gauge) symmetry. As we have seen in chapter 3, the classical equations of motion can then be written in form

$$\begin{aligned} \dot{q}_i &= \{q_i, H_0\}_D \ , \\ \dot{p}_i &= \{p_i, H_0\}_D \ , \\ \Omega^{(2)}_{A_2} &= 0 \ , \end{aligned} \tag{9.1}$$

where the second class constraints are implemented strongly by the Dirac brackets. Since the Dirac brackets have the same algebraic properties as the Poisson brackets, any Dirac bracket of two functions can be computed in the standard way from the fundamental Dirac brackets,

$$\{q_i, p_j\}_D = \{q_i, p_j\} - \sum_{A_2, B_2 = 1}^{N_2} \{q_i, \Omega^{(2)}_{A_2}\} Q^{-1}_{A_2 B_2} \{\Omega^{(2)}_{B_2}, p_j\} \ , \tag{9.2}$$

with

$$Q_{A_2 B_2} = \{\Omega^{(2)}_{A_2}, \Omega^{(2)}_{B_2}\} \ . \tag{9.3}$$

In the case of a non-singular system, the last term on the rhs of (9.2) is absent, and the transition to the quantum theory is effected by replacing $i\hbar\{q_i, p_j\}$ by the commutator of the corresponding operators (denoted by a "hat"), $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$. For a singular system with second class constraints, this prescription is inconsistent with the constraints. Thus consider for example the Poisson bracket of a second class constraint $\Omega^{(2)}_A$ with any function of the canonical variables. This bracket will in general not vanish on the constrained surface, in contrast to the commutator of the corresponding operators, which by construction does. For the same reason one cannot impose the constraints on the states $|\psi>$. On the other hand, since second class constraints are implemented strongly by the Dirac brackets, this suggests that in the presence of such constraints, the fundamental commutators of the canonical variables are obtained from the corresponding Dirac brackets by the prescription

$$[\hat{q}_i, \hat{p}_j] = i\hbar\widehat{\{q_i, p_j\}}_{\mathcal{D}} \ , \tag{9.4}$$

where the hat over the rhs term means, that first the Dirac bracket is computed, and only then the expression so obtained is replaced by the corresponding operator. Actually this prescription is only well defined if the Dirac bracket on the rhs is a non singular expression of the canonical variables, and modulo possible ordering problems. In any case, the quantum version of the rhs of

(9.4) must be defined in such a way, that the second class constraints are implemented strongly. In many cases of physical interest, the transition will be obvious. Modulo such problems, the quantum equations of motion for a purely second class system then take the form of strong equalities,

$$\begin{aligned} i\hbar\dot{\hat{q}}_i &= [\hat{q}_i, \hat{H}] \ , \\ i\hbar\dot{\hat{p}}_i &= [\hat{p}_i, \hat{H}] \ , \\ \hat{\Omega}^{(2)}_{A_2} &= 0 \ , \end{aligned} \tag{9.5}$$

where the commutators are obtained in the usual way from the fundamental commutators (9.4).

9.3 Systems with first and second class constraints

Let us next turn to the case where the Lagrangian also leads to first class constraints $\Omega^{(1)}_{A_1}$, ($A_1 = 1, \cdots, N_1$), i.e., the system exhibits a local symmetry, or "gauge invariance". The quantization of such a system is more subtle. The number of physical degrees of freedom is now further reduced by the number of first class constraints. In this case it is convenient to implement the second class constraints strongly via Dirac brackets and to include *all* first class constraints explicitly in the equations of motion. This does not affect the dynamics of gauge invariant observables. The classical equations of motion then read

$$\begin{aligned} \dot{q}_i &= \{q_i, H\}_{\mathcal{D}} + \xi^{A_1}\{q_i, \Omega^{(1)}_{A_1}\} \ , \\ \dot{p}_i &= \{p_i, H\}_{\mathcal{D}} + \xi^{A_1}\{p_i, \Omega^{(1)}_{A_1}\} \ , \\ \Omega^{(1)}_{A_1} &= 0 \ ; \quad \Omega^{(2)}_{A_2} = 0 \ , \end{aligned} \tag{9.6}$$

where the Dirac brackets are constructed from the second class constraints. Correspondingly, the equation of motion for any function of the canonical variables reads

$$\dot{f}(q,p) \approx \{f, H\}_D + \xi^{A_1}\{f, \Omega^{(1)}_{A_1}\} \ . \tag{9.7}$$

Note that the parameters $\{\xi^{A_1}\}$, reflecting the gauge degrees of freedom, are not fixed by the requirement of persistence in time of the constraints, but are left undetermined. Since the Poisson bracket of any function $f(q,p)$ with a first class constraint is weakly equivalent to the corresponding Dirac bracket, eq. (9.7) can be replaced by

$$\dot{f} \approx \{f, H\}_{\mathcal{D}} + \xi^{A_1}\{f, \Omega^{(1)}_{A_1}\}_{\mathcal{D}} \ , \tag{9.8}$$

so that the second class constraints are now implemented strongly, while the first class constraints continue to be implemented weakly. One would now naively expect that on quantum level the Dirac backets, multiplied by $i\hbar$, are all replaced by the corresponding commutators. While this prescription is consistent with the strong implementation of the second class constraints on operator level, this is not the case for first class constraints. In fact, since $\{f, \Omega^{(1)}_{A_1}\}_{\mathcal{D}}$ does not vanish in general on the subspace $\Omega^{(1)}_{A_1} = 0$, the transition to the commutator is not only inconsistent with implementation of the constraints on operator level, but also inconsistent with their implementation on the states $|\Psi>$, i.e. $\hat{\Omega}^{(1)}_{A_1}|\Psi>= 0$, as seen by considering matrix elements $<\Psi'|[f, \Omega^{(1)}_{A_1}]|\Psi>$. However, as we show below, one can nevertheless formulate an operator valued quantum theory, by introducing a suitable set of subsidary (gauge fixing) conditions, effectively turning the mixed constrained system into a purely second class one. There is however a difference between a purely second class system and a gauge fixed theory. While in the former case all the constraints are part of the dynamical equations generated by a given Lagrangian, the gauge fixing conditions are intoduced from the outside. Gauge variant quantities will of course depend on the choice of gauge, while gauge invariant quantities (observables) do not. This is also evident in (9.8), since in this case the Dirac bracket of $f(q,p)$ with $\Omega^{(1)}_{A_1}$ vanishes weakly, implying independence of the parameters ξ^{A_1}. We now present some details.

Consider the equations of motion (9.6) written entirely in terms of Poisson brackets,

$$\begin{aligned}\dot{q}_i &\approx \{q_i, H^{(1)}\} + \xi^{A_1}\{q_i, \Omega^{(1)}_{A_1}\} ,\\ \dot{p}_i &\approx \{p_i, H^{(1)}\} + \xi^{A_1}\{p_i, \Omega^{(1)}_{A_1}\} ,\end{aligned} \tag{9.9}$$

where $H^{(1)}$ has been defined in (3.65), [1]

$$H^{(1)} = H - \sum_{A_2}^{N_2} \Omega^{(2)}_{A_2} Q^{-1}_{A_2 B_2}\{\Omega^{(2)}_{B_2}, H\} .$$

Since the dynamics of observables on Γ does not depend on the n_1 parameters ξ^{A_1}, we fix these parameters by introducing n_1 suitable (gauge) conditions, where n_1 is the number of first class constraints. Let $\chi_{A_1}(q,p) = 0$ ($A_1 = 1, \cdots, n_1$) be such a set of gauge conditions. If these conditions are to fix the gauge completely, then the vanishing of the variation induced on χ_{A_1} by the first class constraints, i.e.

$$\delta\chi_{A_1} = \epsilon^{B_1}(t)\{\chi_{A_1}, \Omega^{(1)}_{B_1}\} = 0 ,$$

[1] Here H is understood to be weakly equivalent to the canonical Hamiltonian evaluated on the primary surface, H_0.

must necessarily imply the vanishing of the parameters $\{\epsilon^{B_1}\}$. Hence the determinant of the square matrix with elements

$$\Delta_{A_1 B_1} = \{\chi_{A_1}, \Omega^{(1)}_{B_1}\} \tag{9.10}$$

must be different from zero, i.e.,

$$\det\ \Delta \neq 0\,. \tag{9.11}$$

The gauge conditions must therefore be chosen such that (9.11) holds. They must also be consistent with the equations of motion (9.9), augmented by the gauge conditions:

$$\dot{f} = \{f, H^{(1)}\} + \xi^{A_1}\{f, \Omega^{(1)}_{A_1}\}\ , \tag{9.12}$$

$$\Phi_r := (\Omega^{(1)}, \Omega^{(2)}, \chi) = 0\,, \quad r = 1, \cdots, 2n_1 + N_2\ . \tag{9.13}$$

Persistence in time of the subsidiary conditions $\chi_{A_1} = 0$ now requires that [2]

$$\dot{\chi}_{A_1} \approx \{\chi_{A_1}, H^{(1)}\} + \xi^{B_1}\{\chi_{A_1}, \Omega^{(1)}_{B_1}\} \approx 0\ .$$

Because of (9.11), it follows that the gauge parameters are determined:

$$\xi^{A_1} \approx -\sum_{B_1} \Delta^{-1}_{A_1 B_1}\{\chi_{B_1}, H^{(1)}\}\ .$$

We therefore have that

$$\dot{f} \approx \{f, H_{gf}\}\ \ , \tag{9.14}$$

where

$$H_{gf} = H^{(1)} - \sum_{A_1, B_1} \Omega^{(1)}_{A_1}\Delta^{-1}_{A_1 B_1}\{\chi_{B_1}, H^{(1)}\} \tag{9.15}$$

is the gauge fixed Hamiltonian. Hence in the *gauge invariant sector* the equations (9.14) are equivalent to (9.7), with the parameters ξ^{A_1} fixed by the gauge conditions $\chi_{A_1} = 0$. These gauge conditions select from all possible trajectories one representative from each gauge orbit (assuming there exists no Gribov ambiguity [Gribov 1978]). We now prove the following theorem :

Theorem:

In the gauge invariant sector the equations of motion (9.8) are completely equivalent to the set

$$\dot{f} \approx \{f, H\}_{\mathcal{D}^*}\ \ ; \quad \Phi_r = 0\ ,\ \forall r \tag{9.16}$$

[2]The weak equality now includes all the constraints and gauge conditions, $\Phi_r = 0$.

where

$$\{A,B\}_{\mathcal{D}^*} = \{A,B\} - \sum_{r,s}\{A,\Phi_r\}Q^{*-1}_{rs}\{\Phi_s,B\}\ , \tag{9.17}$$

$$Q^*_{rs} = \{\Phi_r,\Phi_s\}\ , \tag{9.18}$$

and

$$\Phi_r := (\chi, \Omega^{(1)}, \Omega^{(2)})\ .$$

Note that now the Dirac bracket involves all the constraints as well as gauge conditions, which together form a second class system.

Proof

We first show that the equations of motion (9.14) are fully equivalent to

$$\dot f = \{f, \bar H_E\}\,, \quad \Phi_r = 0\,, \quad \forall r \tag{9.19}$$

where

$$\bar H_E = H + \eta^{A_2}\Omega^{(2)}_{A_2} + \xi^{A_1}\Omega^{(1)}_{A_1} + \zeta^{A_1}\chi_{A_1} \tag{9.20}$$

is now the fully extended Hamiltonian, which includes *all* constraints and gauge conditions (multiplied by Lagrange multipliers). To prove the above claim we examine the implications of the consistency conditions $\dot\Phi_r = 0$, where the time evolution is now generated by $\bar H_E$:

$$a)\ \dot\Omega^{(1)}_{A_1} \approx \{\Omega^{(1)}_{A_1}, \chi_{B_1}\}\zeta^{B_1} \approx 0\ ,$$

$$b)\ \dot\Omega^{(2)}_{A_2} \approx \{\Omega^{(2)}_{A_2}, H\} + \{\Omega^{(2)}_{A_2}, \Omega^{(2)}_{B_2}\}\eta^{B_2} + \{\Omega^{(2)}_{A_2}, \chi_{B_1}\}\zeta^{B_1} \approx 0\ ,$$

$$c)\ \dot\chi_{A_1} \approx \{\chi_{A_1}, H\} + \{\chi_{A_1}, \Omega^{(2)}_{B_2}\}\eta^{B_2} + \{\chi_{A_1}, \Omega^{(1)}_{B_1}\}\xi^{B_1} + \{\chi_{A_1}, \chi_{B_1}\}\zeta^{B_1} \approx 0\ .$$

In a) we used the fact that the Poisson bracket of $\Omega^{(1)}_{A_1}$ with each of the first three terms on the rhs of (9.20) vanishes on the constrained surface. [3] From a) and (9.11) it follows, that

$$\zeta^{B_1} \approx 0\ . \tag{9.21}$$

Hence the term $\zeta^{A_1}\chi_{A_1}$ in H_E does not contribute to the equations of motion (9.19). It then follows from b) that

$$\eta^{A_2} \approx -\sum_{B_2} Q^{-1}_{A_2B_2}\{\Omega^{(2)}_{B_2}, H\}\ , \tag{9.22}$$

[3] Recall that the constraints $\Omega^{(1)}_{A_1}$ and $\Omega^{(2)}_{A_2}$ were generated by the Dirac algorithm which involves the total Hamiltonian constructed from the canonical Hamiltonian and the primary constraints only. Hence the $\Omega^{(1)}_{A_1}$'s satisfy, in particular, the following equations of motion: $\dot\Omega^{(1)}_{A_1} \approx \{\Omega^{(1)}_{A_1}, H_T\} \approx \{\Omega^{(1)}_{A_1}, H\} \approx 0$.

where $Q_{A_2B_2}$ has been defined in (9.3). With (9.21) and (9.22), c) reduces to

$$\{\chi_{A_1}, H^{(1)}\} + \{\chi_{A_1}, \Omega^{(1)}_{B_1}\}\xi^{B_1} \approx 0 \ ,$$

with the solution

$$\xi^{A_1} \approx -\sum_{B_1} \Delta^{-1}_{A_1B_1}\{\chi_{B_1}, H^{(1)}\} \ , \tag{9.23}$$

where $\Delta_{A_1B_1}$ has been defined in (9.10). Inserting the results (9.21), (9.22), and (9.23) in (9.20), we see that $\bar{H}_E$ can be effectively replaced by (9.15), thus proving the equivalence of (9.14) with (9.19), and therefore also with (9.7) within the gauge invariant sector.

It is now an easy matter to prove the equivalence with (9.16). To this effect we write (9.20) in the compact form

$$\bar{H}_E = H + \rho^r \Phi_r \ , \tag{9.24}$$

where $\rho = (\eta, \chi, \zeta)$. The consistency equations for the constraints and gauge conditions read

$$\dot{\Phi}_r \approx \{\Phi_r, \bar{H}_E\} \approx 0 \ .$$

Since the constraints $\{\Phi_r = 0\}$ form a second class system with $det\{\Phi_r, \Phi_s\} = 0$, we have

$$\rho^r \approx -\sum_{r'} Q^{*-1}_{rr'}\{\Phi_{r'}, H\} \ .$$

Inserting this expression in (9.24) one is immediately led to (9.16), which is now a suitable starting point for making the transition to the quantum theory. Indeed, since all constraints and gauge conditions are now implemented strongly by the $\mathcal{D}^*$-bracket, the transition to the quantum theory is effected by replacing the $\mathcal{D}^*$-bracket, multiplied by $i\hbar$, by the corresponding commutator,

$$[\hat{A}, \hat{B}] = i\hbar\widehat{\{A, B\}}_{\mathcal{D}^*} \ .$$

Clearly the form of the equation of motion will depend on the chosen gauge. On the other hand, the equation of motion for observables, i.e. those functions whose Poisson bracket with the first class constraints vanish weakly, are independent of the choice of gauge.

9.3.1 Example: the free Maxwell field in the Coulomb gauge

We now illustrate the above procedure by an example. Consider once more the free Maxwell field. As we have seen in example 4 of chapter 3, this theory only possesses two first class constraints (one primary and one secondary). In order to fix the gauge completely we therefore need two subsidiary conditions satisfying the requirement (9.11). Since we want to realize the Coulomb gauge, one of these conditions is clearly $\vec{\nabla}\cdot\vec{A}=0$. For the other condition we choose $A^0=0$. [4] This leads to a non-vanishing determinant for the matrix (9.10). Let us label the constraints and gauge conditions as follows:

$$\begin{aligned}\Phi_1 &\equiv \pi_0 = 0\ , \quad \Phi_2 \equiv \partial^j\pi_j = 0\ ,\\ \Phi_3 &\equiv A^0 = 0\ , \quad \Phi_4 \equiv \partial_j A^j = 0\ .\end{aligned}$$

These conditions form a second class system of constraints. The non-vanishing matrix elements $Q^*_{rr'}$ defined in (9.18) are then given by

$$\begin{aligned}Q^*_{13}(\vec{x},\vec{y}) &= -Q^*_{31}(\vec{y},\vec{x}) = -\delta(\vec{x}-\vec{y})\ ,\\ Q^*_{24}(\vec{x},\vec{y}) &= -Q^*_{42}(\vec{y},\vec{x}) = -\nabla^2_x\delta(\vec{x}-\vec{y})\ .\end{aligned}$$

The inverse of the matrix $Q^*_{rr'}(\vec{x},\vec{y})$ is defined by

$$\sum_{r"}\int d^3z\ Q^{*-1}_{rr"}(\vec{x},\vec{z})Q^*_{r"r'}(\vec{z},\vec{y}) = \delta_{rr'}\delta(\vec{x}-\vec{y})\ .$$

The non-vanishing matrix elements of Q^{*-1} are easily calculated to be

$$\begin{aligned}Q^{*-1}_{13}(\vec{x},\vec{y}) &= -Q^{*-1}_{31}(\vec{y},\vec{x}) = \delta(\vec{x}-\vec{y})\ ,\\ Q^{*-1}_{24}(\vec{x},\vec{y}) &= -Q^{*-1}_{42}(\vec{y},\vec{x}) = \frac{1}{\nabla^2}\delta(\vec{x}-\vec{y})\ ,\end{aligned} \tag{9.25}$$

where $\frac{1}{\nabla^2}$ is defined by

$$\frac{1}{\nabla^2}f(\vec{x}) \equiv \int d^3z\ G(\vec{x},\vec{z})f(\vec{z})\ ,$$

with $G(\vec{x},\vec{z})$ the Green function of the Laplace operator,

$$\nabla^2 G(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y})\,.$$

Consider now, e.g., the $\mathcal{D}^*$-bracket (9.17), generalized to a system with an infinite number of degrees of freedom. We have

$$\begin{aligned}\{A^i(\vec{x},t),\pi_j(\vec{y},t)\}_{\mathcal{D}^*} &= \{A^i(\vec{x},t),\pi_j(\vec{y},t)\}\\ &-\sum_{rr'}\int d^3z d^3z'\{A^i(\vec{x},t),\Phi_r(\vec{z},t)\}Q^{*-1}_{rr'}(\vec{z},\vec{z}')\{\Phi_{r'}(\vec{z}'.t),\pi_j(\vec{y},t)\}\ .\end{aligned}$$

[4] On the Lagrangian level these subsidiary conditions would be inconsistent in the presence of sources. Within a Hamiltonian formulation, however, this is an allowed choice, since on the level of the extended Hamiltonian any choice of A^0 can be absorbed by the Lagrange multiplier associated with the secondary (Gauss law) constraint, to be determined by the subsidiary conditions.

Making use of (9.25) one verifies that

$$\{A^i(\vec{x},t), \pi_j(\vec{y},t)\}_{\mathcal{D}^*} = (\delta^i_j - \frac{\partial^i\partial_j}{\nabla^2})\delta(\vec{x}-\vec{y})\,. \tag{9.26}$$

Note that the Coulomb gauge and the secondary constraint $\partial^i\pi_i = 0$ are implemented strongly, as expected. In the same way one finds that

$$\{A^i(\vec{x},t), A^j(\vec{y},t)\}_{\mathcal{D}^*} = 0\,, \quad \{\pi_i(\vec{x},t), \pi_j(\vec{y},t)\}_{\mathcal{D}^*} = 0\,.$$

Furthermore, any $\mathcal{D}^*$-bracket involving A^0 or π_0 vanishes identically. On quantum level, (9.26) translates to

$$[\hat{A}^i(\vec{x},t), \hat{\pi}_j(\vec{y},t)] = i\hbar(\delta^i_j - \frac{\partial^i\partial_j}{\nabla^2})\delta(\vec{x}-\vec{y})\,.$$

This commutator is of course well known, but was constructed here within the Hamiltonian framework. [5]

Summarizing we have: the commutators of the phase-space variables are determined from the $\mathcal{D}^*$-brackets constructed here from the first class constraints and corresponding gauge conditions. These are thereby implemented strongly. Hence the corresponding *operators* can be set equal to the null operator. In general the realization of the operators satisfying the correct commutation relations, as dictated by the $\mathcal{D}^*$-brackets, may however be difficult, if not impossible.

9.3.2 Concluding remark

So far we have considered second class systems, including the case of gauge fixed mixed systems. The basis for their quantization was always provided by their Dirac bracket formulation, where the constraints and gauge conditions were implemented strongly. Apart from ordering problems and possible singularities, this allowed for an operator realization of the equations of motion.

For observables, i.e. gauge invariant quantities, Dirac has proposed an alternative way of dealing with first class or mixed systems on operator level [Dirac 1964]. The first class constraints are imposed as conditions on the physical states in the form [6]

$$\hat{\Omega}^{(1)}_{A_1}|\Psi>= 0\,. \tag{9.27}$$

[5] For less trivial examples see e.g., [Girotti 1982, Kiefer 1985].

[6] If second class constraints are present then we assume that they have been strongly implemented by replacing the Poisson brackets by Dirac brackets constructed from the second class constraints.

The states are therefore also gauge invariant. Hence observables as well as states depend effectively only on gauge invariant combinations of the phase space variables. This ensures on operator level, that for two gauge invariant states $|\Psi>$ and $|\Phi>$

$$< \Psi|[\hat{\mathcal{O}}, \hat{\Omega}^{(1)}_{A_1}]|\Phi >= 0\,, \quad A_1 = 1, \cdots, N_1 \ .$$

In the case of purely first class systems we then have a canonical Poisson bracket structure. In Quantum Mechanics this means that the operators $\hat{\Omega}^{(1)}_{A_1}$ can be realized by making the substitution $p_i \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial q_i}$ in the corresponding classical expression, so that the condition (9.27) takes the form

$$\hat{\Omega}^{(1)}_{A_1}\left(q, \frac{\hbar}{i}\vec{\nabla}\right)\psi(q,t) = 0\ ,$$

where $\psi(q,t)$ is the wave function in configuration space corresponding to the state $|\Psi>$. This is a differential equation, whose solution is the family of all gauge invariant wave functions.

The actual realization of Dirac's method may however not be straightforward. Thus the definition of the operators $\hat{\Omega}^{(1)}_{A_1}$ may be problematic due to ordering problems. Furthermore one in general also seeks to preserve the classical algebraic properties of the constraints, which may also be problematic.

Another method consists in quantizing only gauge invariant degrees of freedom constructed from the q_i's and p_i's, and all observables are expressed in terms of these. Hence no gauge conditions are imposed. This reduced phase space is endowed with the standard Poisson bracket structure. For a systematic method of arriving at such a reduced phase space formulation of the dynamics we refer the reader to the last section of chapter 4. Correlation functions of observables (which only depend on the star-variables) will have the standard path integral representation for an unconstrained system. But the price paid for such a formulation may be too high to be useful. For example, in a gauge theory like the Maxwell theory, the star-variables are given by the *non-local* (gauge invariant) expressions for the transverse potentials and their corresponding conjugate momenta. It is therefore of interest to develop non-operator methods based on functional techniques. This will be the subject of the following three chapters.