PH 6140: Quantum Yang-Mills Theory

Lecture 7: Notes on path integral quantization of Maxwell field theory

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We recall that for a set of constrained degrees of freedom (gauge fields), $\{\varphi_i\}$ the path integrals according to the Faddeev-Popov ansatz is,

$$Z = \mathcal{N} \int \prod_i [d\varphi_i] \; e^{iI[\{\varphi_i\}]} \, \delta\left(F(x)\right) \left| \frac{\delta F(x)}{\delta \alpha(y)} \right|_{F=0}$$

where the delta function enforces the gauge fixing condition is,

$$F\left(\{\varphi_i\}\right) = 0$$

and the corresponding gauge fixing determinant is,

$$\left| \frac{\delta F(x)}{\delta \alpha(y)} \right|_{F=0}$$

with $\alpha(x)$ being the gauge transformation parameter. Here we will use this Faddeev-Popov ansatz and quantize the Maxwell field theory in Coulomb and Lorenz gauge. The Maxwell field theory is defined by the action,

$$I\left[A_\mu\right] = \int d^4x \left(-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right), F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha.$$

This theory has a gauge symmetry,

$$A_\mu \to A'_\mu \to A_\mu + \partial_\mu \alpha(x),$$

the symmetry parameter, $\alpha(x)$ being an *arbitrary* function of spacetime.

1 Coulomb gauge

Coulomb gauge is a full gauge fix, i.e. it removes all redundant degrees of freedom, namely the temporal and longitudinal polarization and keeps only the *physical* degrees of freedom, namely the transverse polarizations. We will see how to implement this in the path integral via Faddeev-Popov ansatz. The Coulomb gauge is given by *two* gauge fixing conditions, namely

$$F_1(x) = A^0(x) = 0,$$

$$F_2(x) = \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0.$$

So the Faddeev-Popov ansatz for the path integral in the Coulomb gauge is,

$$Z = \mathcal{N} \int \prod_\mu [dA_\mu] \; e^{iI[A_\mu]} \, \delta\left(F_1(x)\right) \left| \frac{\delta F_1(x)}{\delta \alpha(y)} \right|_{F_1=0} \delta\left(F_2(x)\right) \left| \frac{\delta F_2(x)}{\delta \alpha(y)} \right|_{F_2=0}$$

To make progress with evaluating the path integral we first have to determine the gauge fixing determinants. Under a gauge transformation by a parameter $\alpha(x)$,

$$F_1(x) \to F_1^{\alpha}(x) = F_1(x) + \partial_0 \alpha(x)$$
$$\Rightarrow \frac{\delta F_1^{\alpha}(x)}{\delta \alpha(y)} = \partial_0 \delta^4(x-y).$$

Similarly, one can show,

$$\frac{\delta F_2^{\alpha}(x)}{\delta \alpha(y)} = \boldsymbol{\nabla}^2 \delta^4(x-y).$$

So for both gauge fixing determinants aka Jacobians are independent of $\alpha(x)$ or $A_\mu(x)$ (which means these can be taken out the of the path integral as they are independent of the integration variable)! So the path integral is,

$$\begin{aligned} Z &= \mathcal{N} \int \prod_\mu [dA_\mu]\ e^{iI[A_\mu]}\, \delta\left(A^0\right)\ \delta\left(\boldsymbol{\nabla} \cdot \boldsymbol{A}\right)\ \partial_0 \delta^4(x-y)\ \boldsymbol{\nabla}^2 \delta^4(x-y) \\ &= \mathcal{N}' \int \prod_\mu [dA_\mu]\ e^{iI[A_\mu]}\, \delta\left(A^0\right)\ \delta\left(\boldsymbol{\nabla} \cdot \boldsymbol{A}\right). \end{aligned} \tag{1}$$

In the second step we have taken the gauge fixing determinants out of the integral and absorbed them into the normalization constant, $\mathcal{N}'$.

Recall that the action is,

$$\begin{aligned} I\left[A_\mu\right] &= \int d^4x \left(-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}\right), F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \\ &= -\frac{1}{2} \int d^4x\ \left(\partial_\alpha A_\beta\right)\left(\partial^\alpha A^\beta\right) + \frac{1}{2} \int d^4x\ \left(\partial_\alpha A_\beta\right)\left(\partial^\beta A^\alpha\right) \\ &= -\frac{1}{2} \int d^4x\ \left(\partial_\alpha A_\beta\right)\left(\partial^\alpha A^\beta\right) + \frac{1}{2} \int d^4x\ \left(\partial_\alpha A^\alpha\right)\left(\partial^\beta A_\beta\right) + \underbrace{\int d^4x\ (\text{total derivative terms})}_{\to 0} \\ &= -\frac{1}{2} \int d^4x\ \left[\left(\partial_\alpha A^0\right)\left(\partial^\alpha A^0\right) + \left(\partial_\alpha \mathbf{A}\right) \cdot \left(\partial^\alpha \mathbf{A}\right) - \left(\dot{A}^0 + \boldsymbol{\nabla} \cdot \boldsymbol{A}\right)^2\right] \end{aligned}$$

Using this expression for the action in the path integral (1), one has,

$$Z = \mathcal{N}' \int \prod_\mu [dA_\mu] \exp\left(-\frac{i}{2} \int d^4x\ \left[\left(\partial_\alpha A^0\right)\left(\partial^\alpha A^0\right) + \left(\partial_\alpha \mathbf{A}\right) \cdot \left(\partial^\alpha \mathbf{A}\right) - \left(\dot{A}^0 + \boldsymbol{\nabla} \cdot \boldsymbol{A}\right)^2\right]\right) \delta\left(A^0\right) \delta\left(\boldsymbol{\nabla} \cdot \boldsymbol{A}\right)$$

The A_0 path integration can be readily done due to the delta function $\delta(A^0)$, and then one is left with,

$$Z = \mathcal{N}' \int \left[d^3\mathbf{A}\right]\ \exp\left(-\frac{i}{2} \int d^4x\ \left[\left(\partial_\alpha \mathbf{A}\right) \cdot \left(\partial^\alpha \mathbf{A}\right) - \left(\boldsymbol{\nabla} \cdot \boldsymbol{A}\right)^2\right]\right)\ \delta\left(\boldsymbol{\nabla} \cdot \boldsymbol{A}\right) \tag{2}$$

To make further progress one makes a change of variables in the path integral variables, i.e. $\mathbf{A}$. Instead of resolving $\mathbf{A}$ into its Cartesian components A_1, A_2, A_3, we split up $\mathbf{A}$ into a longitudinal part, namely, $\mathbf{A}_\parallel$ and a transverse part, namely, $\mathbf{A}_\perp$ in the usual manner,

$$\mathbf{A} = \mathbf{A}_\parallel + \mathbf{A}_\perp,$$

with,

$$\boldsymbol{\nabla} \cdot \mathbf{A}_\perp = 0, \qquad \boldsymbol{\nabla} \times \mathbf{A}_\parallel = 0.$$

The terminology (longitudinal and transverse) become clear when one moves to momentum/ Fourier space whereby these conditions look like,

$$\hat{\boldsymbol{k}} \cdot \mathbf{A}_\perp = 0, \qquad \hat{\boldsymbol{k}} \times \mathbf{A}_\parallel = 0.$$

Due to these equations, one can check that the combined number of degrees of freedom of $\mathbf{A}_{\parallel}, \mathbf{A}_{\perp}$ is still 3 i.e. same as the original $\mathbf{A}$. Thus due to the change of variables we have the measure of the path integral,

$$\left[d^3\mathbf{A}\right] = \left[d\mathbf{A}_{\parallel}\right]\left[d\mathbf{A}_{\perp}\right]$$

while the term in the action,

$$\left(\partial_\alpha \mathbf{A}\right)\cdot\left(\partial^\alpha \mathbf{A}\right) = \left(\partial_\alpha \mathbf{A}_{\perp}\right)\cdot\left(\partial^\alpha \mathbf{A}_{\perp}\right) + 2\left(\partial_\alpha \mathbf{A}_{\perp}\right)\cdot\left(\partial^\alpha \mathbf{A}_{\parallel}\right) + \left(\partial_\alpha \mathbf{A}_{\parallel}\right)\cdot\left(\partial^\alpha \mathbf{A}_{\parallel}\right),$$

and the term,

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \boldsymbol{\nabla}\cdot\boldsymbol{A}_{\parallel}.$$

Note that $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ implies in Fourier space $\mathbf{k}\cdot\mathbf{A} = |\mathbf{k}||A_{\parallel}| = 0$[1]. Now we plug back everything in the path integral expression (2) and rewrite it in Fourier space

$$Z = \mathcal{N}'\int\left[d\mathbf{A}_{\parallel}(x)\right]\left[d\mathbf{A}_{\perp}(x)\right]\exp\left(-\frac{i}{2}\int d^4x\,\left[\left(\partial_\alpha \mathbf{A}_{\perp}\right)\cdot\left(\partial^\alpha \mathbf{A}_{\perp}\right) + 2\left(\partial_\alpha \mathbf{A}_{\perp}\right)\cdot\left(\partial^\alpha \mathbf{A}_{\parallel}\right) + \left(\partial_\alpha \mathbf{A}_{\parallel}\right)\cdot\left(\partial^\alpha \mathbf{A}_{\parallel}\right) - \left(\boldsymbol{\nabla}\cdot\mathbf{A}_{\parallel}\right)^2\right]\right)\delta\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}_{\parallel}\right)$$

$$= \mathcal{N}'\int\left[d\mathbf{A}_{\parallel}(k)\right]\left[d\mathbf{A}_{\perp}(k)\right]$$

$$\exp\left(-\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\,\left[\left(ik_\alpha \mathbf{A}_{\perp}\right)\cdot\left(-ik^\alpha \mathbf{A}_{\perp}\right) + 2\left(ik_\alpha \mathbf{A}_{\perp}\right)\cdot\left(-ik^\alpha \mathbf{A}_{\parallel}\right) + \left(ik_\alpha \mathbf{A}_{\parallel}\right)\cdot\left(-ik^\alpha \mathbf{A}_{\parallel}\right) + \left(|\mathbf{k}||A_{\parallel}|\right)^2\right]\right)\frac{\delta\left(|A_{\parallel}|\right)}{|\mathbf{k}|}$$

Now one can easily do the $\mathbf{A}_{\parallel}$ integration to set to zero all terms involving $\mathbf{A}_{\parallel}$ and then move back to position space to obtain,

$$Z = \mathcal{N}''\int\left[d\mathbf{A}_{\perp}\right]\,\exp\left(-\frac{i}{2}\int d^4x\,\left[\left(\partial_\alpha \mathbf{A}_{\perp}\right)\cdot\left(\partial^\alpha \mathbf{A}_{\perp}\right)\right]\right). \tag{3}$$

Thus in the final expression for the path integral, one only has physical transverse modes of the Maxwell field, $\mathbf{A}_{\perp}$. This is guaranteed to produce the Coulomb gauge propagator $\langle T\left(A_i(x)A_j(y)\right)\rangle$ obtained previously using Dirac method.

Homework 1: Check that you indeed reproduce the Coulomb gauge propagator by computing the path integral! *Hint*: It is best to work in Fourier space and read off the Feynman rule/factor for the propagator i.e. $\frac{1}{2}A^{-1}_{ij}$ from the path integral (3).

2 Lorenz Gauge

Next we work in Lorenz gauge, $F = \partial_\mu A^\mu = 0$, which we already know is only a partial/incomplete gauge fix. The Faddeev-Popov ansatz for this case is,

$$Z = \mathcal{N}\int\prod_\mu\left[dA_\mu\right]\,e^{iI[A_\mu]}\,\delta\left(F(x)\right)\left|\frac{\delta F(x)}{\delta\alpha(y)}\right|_{F=0}$$

Under a gauge transformation by a gauge parameter $\alpha(x)$,

$$\begin{aligned} F(x) \to F^\alpha(x) &= \partial_\mu\left(A^\mu + \partial^\mu\alpha\right) \\ &= F(x) + \Box\alpha(x) \\ \Rightarrow \frac{\delta F^\alpha(x)}{\delta\alpha(y)} &= \Box\delta^4(x-y). \end{aligned}$$

[1]Also, recall that in Fourier space the transverse component is expressed as

$$\begin{aligned} \mathbf{A}_{\perp} &= \mathbf{A} - \mathbf{A}_{\parallel} \\ &= \mathbf{A}(\mathbf{k}) - \frac{\left(\mathbf{A}\cdot\mathbf{k}\right)\mathbf{k}}{\mathbf{k}\cdot\mathbf{k}} \\ \Rightarrow A^i_{\perp} &= \left(\delta_{ij} - \frac{k_ik_j}{\mathbf{k}^2}\right)A_j. \end{aligned}$$

Thus *again* in this case the gauge fixing determinant is independent of the gauge field, A_μ. In fact this is not at all surprising and this will always be so whenever the gauge fixing condition is a linear function of A_μ. Since the gauge fixing determinant is independent of the path integration variable A_μ, just as in the case of Coulomb case we will take it out of the path integral and absorb it into the normalization constant, $\mathcal{N}$. Thus, the path integral now looks like,

$$\begin{aligned}
Z &= \mathcal{N}' \int \prod_\mu [dA_\mu]\ e^{iI[A_\mu]}\, \delta\left(\partial_\mu A^\mu\right) \\
&= \mathcal{N}' \int \prod_\mu [dA_\mu]\ \exp\left[-\frac{i}{2}\int d^4x\ \left(\partial_\alpha A_\beta\right)\left(\partial^\alpha A^\beta\right) + \frac{i}{2}\int d^4x\ \left(\partial_\alpha A^\alpha\right)\left(\partial^\beta A_\beta\right)\right]\ \delta\left(\partial_\mu A^\mu\right) \\
&= \mathcal{N}' \int \prod_\mu [dA_\mu]\ \exp\left[-\frac{1}{2}\int d^4x\ \left(\partial_\alpha A_\beta\right)\left(\partial^\alpha A^\beta\right)\right]\ \delta\left(\partial_\mu A^\mu\right)
\end{aligned} \tag{4}$$

However we cannot make any further progress as one cannot get rid of the delta function by performing an integral over $\partial_\mu A^\mu$ and it is also not possible to split A^μ into components parallel to k^μ and orthogonal to k^μ as we are dealing with 4 vectors instead of 3 vectors. So to make progress we will adopt a different route/ alter our starting point i.e. Lorenz gauge condition. We work with a modified gauge condition,

$$\partial_\mu A^\mu = c(x).$$

Here $c(x)$ is a Lorentz scalar. This is still a Lorentz invariant gauge condition, just as Lorenz gauge is. Given this gauge fixing condition,

$$G(x) = \partial_\mu A^\mu(x) - c(x) = 0.$$

Clearly the gauge fixing determinant/ Jacobian remains unchanged, same as that in Lorenz gauge,

$$\frac{\delta G^\alpha(x)}{\delta\alpha(y)} = \Box\delta^4(x-y).$$

So the path integral is now,

$$\begin{aligned}
Z &= \mathcal{N} \int \prod_\mu [dA_\mu]\ e^{iI[A_\mu]}\, \delta\left(G(x)\right) \left|\frac{\delta G(x)}{\delta\alpha(y)}\right|_{F=0} \\
&= \mathcal{N} \int \prod_\mu [dA_\mu]\ e^{iI[A_\mu]}\, \delta\left(\partial_\mu A^\mu - c\right) \Box\delta^4(x-y) \\
&= \mathcal{N}' \int \prod_\mu [dA_\mu]\ e^{iI[A_\mu]}\, \delta\left(\partial_\mu A^\mu - c\right).
\end{aligned} \tag{5}$$

In the last we have taken the gauge fixing determinant out of the path integral as it is independent of the path integration variable, A_μ. Next comes the crucial step, we will ***average over all possible functions***, $c(x)$ with weight peaked symmetrically around $c(x) = 0$, to be specific a Gaussian weight,

$$\mathcal{N}_c e^{-\frac{i\lambda}{2}\int d^4x\ c^2(x)}$$

where $\mathcal{N}_c$ is a normalization factor for the Gaussian distribution. The parameter λ which represents the (inverse) standard deviation/ spread of the Gaussian distribution and is completely arbitrary. Why is this justified? This is justified because if the theory is gauge-invariant, then the path integrals expression for the usual Lorenz gauge Eq. (4) and the path integral expression for the more general Lorentz invariant gauge (5) must be the same, since they represent the same theory. Thus averaging over all Lorentz invariant gauges functions $c(x)$ should produce the ***same*** result.

$$Z' = \mathcal{N}_c \int [dc] e^{-\frac{i\lambda}{2}\int d^4x\ c^2(x)} \left(\mathcal{N}' \int \prod_\mu [dA_\mu]\ e^{iI[A_\mu]}\, \delta\left(\partial_\mu A^\mu - c\right)\right).$$

Switching order of integration,

$$Z' = \mathcal{N}'\mathcal{N}_c \int \prod_\mu [dA_\mu]\; e^{iI[A_\mu]} \left(\int [dc] e^{-\frac{1\lambda}{2}\int d^4x\, c^2(x)}\, \delta\left(\partial_\mu A^\mu - c\right) \right)$$
$$= \mathcal{N}'' \int \prod_\mu [dA_\mu]\; e^{iI'[A_\mu]}, \qquad I'[A_\mu] = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}\left(\partial_\mu A^\mu\right)^2 \right). \qquad (6)$$

Thus, we have succeeded in getting rid of the gauge fixing delta function! However the cost we have to pay is that we have a modified action, $I'[A_\mu]$, which is highly ambiguous to the presence of an *arbitrary* constant parameter, λ.

Another equivalent way of doing things to observe that since $c(x)$ is a Lorentz invariant gauge function, the path integral result (5), whatever if might be after dealing with the delta function, should be independent of $c(x)$. After observing this we multiply the path integral (5) by a unity, by means of a normalized Gaussian path integral, namely,

$$\mathcal{N}_c \int [dc] e^{-\frac{i\lambda}{2}\int d^4x\, c^2(x)} = 1.$$

Then,

$$Z = (1) \times \mathcal{N}' \int \prod_\mu [dA_\mu]\; e^{iI[A_\mu]}\, \delta\left(\partial_\mu A^\mu - c\right)$$
$$= \left(\mathcal{N}_c \int [dc] e^{-\frac{i\lambda}{2}\int d^4x\, c^2(x)} \right) \times \mathcal{N}' \int \prod_\mu [dA_\mu]\; e^{iI[A_\mu]}\, \delta\left(\partial_\mu A^\mu - c\right)$$
$$= \mathcal{N}'\mathcal{N}_c \int \prod_\mu [dA_\mu]\; e^{iI[A_\mu]} \left(\int [dc] e^{-\frac{1\lambda}{2}\int d^4x\, c^2(x)}\, \delta\left(\partial_\mu A^\mu - c\right) \right)$$
$$= \mathcal{N}'' \int \prod_\mu [dA_\mu]\; e^{iI'[A_\mu]}, \qquad I'[A_\mu] = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}\left(\partial_\mu A^\mu\right)^2 \right).$$

In class I followed this second way of deriving the Lorenz gauge Maxwell path integral.

Homework 2: From the Lorentz gauge path integral for the Maxwell field (6), deduce the Feynman propagator of the Maxwell field in Lorentz gauge, namely, $\langle T\left(A_\mu(x)A_\nu(y)\right)\rangle$. You should get the same expression as you did in the Gupta-Bleuler covariant quantization of the Maxwell field. *Hint*: Going to Fourier space will make life easier.

Homework 3: Follow the Noether method to derive the expression for the Noether currents for complex scalar n-tuple field theory described by the action,

$$I[\Phi] = \int d^4x \left[\left(\partial^\mu \Phi\right)^\dagger \left(\partial_\mu \Phi\right) - m^2 \Phi^\dagger \Phi - V(\Phi^\dagger \Phi) \right]$$

which is invariant under global $U(N)$ symmetry:

$$\Phi \rightarrow \Phi' = U\,\Phi,$$

where U is an $n \times n$ unitary matrix which can be expressed in the exponential form in terms of its hermitian generators, T_a, $a = 1, ..., N^2$, as

$$U = \exp\left(i\alpha^a T_a\right).$$

α^a's are the continuous symmetry parameters, one corresponding to each generator, T_a. *Hint:* The final expression has already been presented in the lecture, $J_a^\mu = i\left(\partial^\mu \Phi^\dagger T_a \Phi - \Phi^\dagger T_a \partial^\mu \Phi\right)$.