Quantum Yang Mills (PH 6140): Lecture 8 & 9
Non-Abelian Gauge Symmmetry

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Let's consider a theory in which the degrees of freedom are N complex scalar fields, $\Phi_i,\ i=1,2,\ldots,N$. If they are all independent, then the action would be,

$$I\left[\{\Phi_i(x)\}\right]=\int d^4x\,\sum_i\left[(\partial^\mu\Phi_i)^*\,\partial_\mu\Phi_i-m_i^2\Phi_i^*\Phi_i-V^i\left(\Phi_i^*\Phi_i\right)\right].$$

In particular the global $U(1)$ symmetry of a single complex scalar field is is now turned into a symmetry group which is a direct product (multiple copies) of N independent $U(1)$ factors, i.e. $U(1)\times U(1)\times\ldots\times U(1)=U(1)^N$,

$$\Phi_i\to\Phi_i'=e^{i\,e_i\alpha^i}\Phi_i,$$

where the i-th scalar has charge e_i. Following the *gauge principle* we can turn this global $U(1)^N$ symmetry into a local/gauge symmetry by introducing a gauge field for each scalar, a different Maxwell field, $A^i_\mu(x)$, and then replacing the respective partial derivative by the covariant derivative,

$$\partial_\mu\Phi_i\to D_\mu\Phi_i=\partial_\mu\Phi_i+i\,e\,A^i_\mu\Phi_i.$$

The combined gauge transformation for all sets of fields is then,

$$\Phi^i\to\Phi'^i=e^{i\,e_i\alpha^i(x)}\Phi$$
$$A^i_\mu\to\left(A'_\mu\right)^i=A^i_\mu+\partial_\mu\alpha^i(x)$$

But this system does not give rise to any new phenomenon compared to the case of a single complex scalar (this system is just a collection of N decoupled scalar fields with their own/individual Maxwell field). However, some new interesting physics emerges when we consider the special case when the mass parameters of the scalar fields become identical, $m_i=m,\forall\,i$. Especially consider the case when the scalars are free i.e. $V_i=0,\forall\,i$. It is clear that in this case all the scalars can be gathered together in a column vector, $\mathbf{\Phi} = \begin{pmatrix} \Phi^1 \\ \Phi^2 \\ \vdots \\ \Phi^N \end{pmatrix}$ and the action can be expressed in a very compact form,

$$I\left[\{\Phi_i(x)\}\right] = \int d^4x \, \left[\left(\partial^\mu \mathbf{\Phi}\right)^\dagger \partial_\mu \mathbf{\Phi} - m_i^2 \mathbf{\Phi}^\dagger \mathbf{\Phi} \right].$$

It is now evident that the action is now invariant under a larger group/matrix transformation, namely

$$\mathbf{\Phi} \to \mathbf{\Phi}' = U\mathbf{\Phi},$$

where, $U \in U(N)$, i.e. $N \times N$ rank unitary matrices. One can now add a potential term which will respect this unitary symmetry of the free theory, namely,

$$I\left[\mathbf{\Phi}\right] = \int d^4x \left[\left(\partial^\mu \mathbf{\Phi}^\dagger\right) \left(\partial_\mu \mathbf{\Phi}\right) - m^2 \mathbf{\Phi}^\dagger \mathbf{\Phi} - V(\mathbf{\Phi}^\dagger \mathbf{\Phi}) \right]. \tag{1}$$

Compared to the previous case we see that not only are the mass parameters are identical now, but their are interactions between the different scalar components, Φ_i's. Just like the $U(1)$ case here the set of all possible $U(N)$ symmetry transformations form a group under the operation of matrix multiplication. (One can convince oneself easily that product of two unitary matrices, U_1 and U_2 is also unitary, identity matrix, I_N is also unitary and the inverse of an unitary matrix is also unitary). However compared to the $U(1)$ case there is a crucial distinction - two successive distinct U(N) symmetry transformations do not commute in general since matrix multiplication is in general non-commutative,

$$U_1 U_2 \neq U_2 U_1$$

In general, such symmetry groups are dubbed *Nonabelian* symmetry groups. Another example of an nonabelian symmetry group is the group of rotations in three space dimensions, namely, $SO(3)$ - we are familiar with the fact that final configuration of a system after two successive rotations around distinct axes differs depends on the sequence in which the two rotations are carried out.

1 Continuous Groups, Lie Groups, Generators, Lie Algebras

Recall that a global $U(1)$ transformation is multiplication by a phase factor

$$U(\alpha) = e^{ie\alpha},$$

where e is a quantum of the charge of the scalar field and α is the symmetry parameter. The symmetry parameter is a continuous variable taking any real value i.e. $\alpha \in \mathbb{R}$.

Since α is a continuous variable, the group elements, $U(\alpha)$, being a function of a continuous variable is itself continuous. Such groups whose group elements are continuous functions of some real valued variables (in this case the symmetry parameter α) are called *continuous groups.* They contain infinite number of elements. In addition this group ($U(1)$) is a *Lie Group*, i.e. the group elements are an *analytic* functions of the symmetry parameter α. This is evident because the exponential can be expanded in a power series,

$$U(\alpha) = e^{ie\alpha} = \sum_n \frac{(ie\alpha)^n}{n!}.$$

These ideas carry forward to the nonabelian case as well, let's say when the symmetry group is $U(N)$. One is familiar with the fact that any $N \times N$ unitary matrix, U, can be expressed as exponential of an $N \times N$ hermitian matrix, H,

$$U = e^{iH}.$$

Thus the space of all possible $N \times N$ unitary matrices is obtained from the vector space of all possible $N \times N$ hermitian matrices. Further one can expand any $N \times N$ hermitian matrix, H in a basis of N^2 hermitian matrices, namely, $T_a, a = 1, ..., N^2$,

$$H = \alpha^a \, T_a$$

where the α_a's are the basis expansion coefficients and they can each take real values, i.e. $\alpha_a \in \mathbb{R}$. We are following the Einstein summation convention whereby repeated indices are summed over. The unitary group element is thus expressed in the form,

$$U = \exp\left(i\alpha^a \, T_a\right).$$

The basis matrices, T_a's, are called the *generators* of the Lie group. The total number of generators is the dimension of the Lie algebra. In the present case, i.e. the Lie algebra $u(N)$, the dimension is N^2. (Although we are discussing at present the $U(N)$ group the discussion extends to all Lie groups). We can identify the α^a's as the continuous symmetry parameters while the hermitian matrix basis elements T_a's as the counterparts for charge quantum, e. Just as different values of e represent different particles (representations of the $U(1)$ group), different representations i.e. matrices of different sizes correspond to different non-abelian charges/particles. In particular, the scalar $\mathbf{\Phi}$ transforms in the *fundamental* or defining representation of the $u(N)$ i.e. it gets multiplied from the left by an $N \times N$ unitary matrix under the symmetry,

$$\mathbf{\Phi} \to \mathbf{\Phi}' = U \, \mathbf{\Phi}$$

Thus for each continuous symmetry parameter, α^a, there is a conserved Noether current, J^μ_a. It is evident from the Noether current expression computed from the action,

$$J^\mu{}_a = i\left(\partial^\mu \mathbf{\Phi}^\dagger T_a \mathbf{\Phi} - \mathbf{\Phi}^\dagger T_a \partial^\mu \mathbf{\Phi}\right), \; a = 1, ..., N^2.$$

An important consequence of closure under the matrix multiplication of matrices, $U = \exp(i\alpha^a T_a)$ i.e. demanding that,

$$\exp\left(i\alpha^a T_a\right) \exp\left(i\alpha^b T_b\right) = \exp\left(i\alpha^c T_c\right)$$

is that the T_a's must obey a commutation relation of the type,

$$[T_a, T_b] = i f^c{}_{ab} T_c. \tag{2}$$

Such an algebra of the generators of a Lie group is called the *Lie algebra* of the group, denoted by a lower case letter, $u(N)$. From the defining equation, (2), it is evident that the constants, f^c_{ab} are real and antisymmetric in the lower pair of indices, ab. These constants are called the *structure constants* of the Lie algebra. Another important property of a Lie algebra is the Jacobi identity:

$$[[T_a, T_b], T_c] + [[T_b, T_c], T_a] + [[T_c, T_a], T_b] = 0. \tag{3}$$

In terms of the structure constants the Jacobi identity is expressed as,

$$f^d{}_{ab} f^e{}_{cd} + f^d{}_{bc} f^e{}_{ad} + f^d{}_{ca} f^e{}_{bd} = 0. \tag{4}$$

1.1 Adjoint Representation

The structure constants of a Lie algebra themselves provide a representation is the *adjoint* or *regular* representation which is given by the explicit form of the matrix elements of the generators,

$$\left(T_b^{\mathrm{Adj}}\right)^a{}_c = i f^a{}_{bc} \tag{5}$$

Here the upper left index is a row index and the bottom right index is a column index. It can easily be verified that these elements defined by (5) satisfy the relations (2) and (3). Since the row and column index go over the dimension of the Lie algebra, the generators of the Lie algebra in the adjoint representation are $N^2 \times N^2$ matrices for the case of $u(N)$. For a general nonabelian Lie group, the adjoint representation are $dim(g) \times dim(g)$ matrices, where $dim(g)$ is the dimension of the lie algebra.

A field in the adjoint representation, say V transforms as,

$$V' = \exp\left(i\alpha^a T_a^{\mathrm{Adj}}\right) V \approx \left(\mathbb{I} + i\alpha^a T_a^{\mathrm{Adj}}\right) V + O\left(\alpha^2\right).$$

Displaying the indices this infinitesimal version equation looks like,

$$\begin{aligned} V'^a &= \left(\delta^a_c + i\alpha^b \left(T_b^{\mathrm{Adj}}\right)^a{}_c\right) V^c \\ &= V^a + i\alpha^b \left(i f^a{}_{bc}\right) V^c \\ &= V^a + f^a{}_{bc} V^b \alpha^c. \end{aligned} \tag{6}$$

2 Nonabelian gauge symmetry

We have seen in the complex scalar case which has an abelian $U(1)$ global symmetry, turning the global symmetry $U(1)$ into a local/gauge symmetry using the gauge principle or the principle of minimal coupling leads to coupling of the complex scalar to some form of massless vector field (Maxwell) . Here we try to do this for the nonabelian symmetry, $U(N)$ i.e. attempt to turn the $U(N)$ into local (gauge symmetry) and see what kinda massless vector field does this N-component scalar couple to. For the scalar electrodynamics case we this covariant derivative can be defined using a minimal coupling form by introducing a vector field A_μ,

$$D_\mu \mathbf{\Phi} \equiv (\partial_\mu - i\, e\, A_\mu)\, \mathbf{\Phi}.$$

However we have witnessed in the previous section that for nonabelian symmetry, the charge e must be replaced by the generator T_a. Hence balancing indices one can propose a gauge covariant derivative for a nonabelian gauge symmetry to be of the form,

$$D_\mu \mathbf{\Phi} = \left(\partial_\mu - iT_a\, A^a_\mu\right) \mathbf{\Phi}, \tag{7}$$

i.e. corresponding to each generator T_a, we introduce a vector (spin 1) gauge field with the same index, A^a_μ. Then, covariance of this new derivative under the local $U(N)$ which deliver us the transformation law for the gauge field(s) as follows, (to reduce cumber we stop displaying the functional dependence of $\mathbf{\Phi}$ and U on x),

$$\begin{aligned} D'_\mu \mathbf{\Phi}' &= U\, D_\mu \mathbf{\Phi} \\ \Rightarrow \left(\partial_\mu - iT_a\, A'^a_\mu\right) (U\, \mathbf{\Phi}) &= U\, \left(\partial_\mu - T_a\, A^a_\mu\right) \mathbf{\Phi} \\ \Rightarrow \partial_\mu\, (U\, \mathbf{\Phi}) - iT_a\, A'^a_\mu U\, \mathbf{\Phi} &= U \partial_\mu \mathbf{\Phi} - iU\, T_a\, A^a_\mu\, \mathbf{\Phi} \\ \Rightarrow (\partial_\mu U)\ \mathbf{\Phi} - iT_a\, A'^a_\mu\, U\, \mathbf{\Phi} &= -i\, U\, T_a\, A^a_\mu\, \mathbf{\Phi} \\ \Rightarrow iT_a\, A'^a_\mu\, U\, \mathbf{\Phi} &= iU\, T_a\, A^a_\mu \mathbf{\Phi} + (\partial_\mu U)\ \mathbf{\Phi} \\ \Rightarrow iT_a\, A'^a_\mu U\, \mathbf{\Phi} &= \left(i\, U\, T_a\, A^a_\mu\, U^{-1}\right) U\, \mathbf{\Phi} + (\partial_\mu U)\ U^{-1}\, U\, \mathbf{\Phi}. \end{aligned}$$

Since this holds for arbitrary $\mathbf{\Phi}$ and U, we must have,

$$i\, T_a\, A'^a_\mu = i\, U\, T_a\, A_\mu\, U^{-1} + (\partial_\mu U)\ U^{-1}$$

which upon a bit of algebraic simplification gives,

$$T_a\, A'^a_\mu = U\, T_a\, A^a_\mu\, U^{-1} - i\, (\partial_\mu U)\ U^{-1}. \tag{8}$$

Thus the matrix-valued vector field $\boldsymbol{A}_\mu \equiv T_a\, A^a_\mu$ introduced for the case of $U((N)$ or $SU(N)$ is a generalization of the 4-vector potential of electrodynamics (aka the Maxwell field) and is referred to as a *non-abelian gauge field*. Specifically for the case when the symmetry is $SU(N)$, i.e. the group of unitary matrices with unit determinant, the gauge field is called Yang-Mills gauge field. in index free notation, the transformation law for the gauge field is expressed as,

$$\boldsymbol{A}'_\mu = U \boldsymbol{A}_\mu U^{-1} - i \left(\partial_\mu U\right) U^{-1}. \tag{9}$$

It in instructive to consider the special case when the gauge transformation is close to identity, i.e. the *infinitesimal* form,

$$U \approx \mathbb{I} + i\alpha^a T_a.$$

Substituting in (8), one has the infinitesimal version of the transformation law of the gauge field component,

$$A'^a_\mu = A^a_\mu + \partial_\mu \alpha^a + f^a{}_{bc} A^b_\mu \alpha^c + O(\alpha^2). \tag{10}$$

Comparing this expression to that of the abelian $U(1)$ case, i.e. for the Maxwell field, namely

$$A'_\mu = A_\mu + \partial_\mu \alpha,$$

we observe that in this expression for the infinitesimal form of the nonabelian gauge field, one has, in addition to the derivative of α term, another piece proportional to the structure constants of the Lie group. What is the physical significance of this piece? Notice that in the special case when the symmetry goes global, i.e. α^a's are constants, i.e. they take the same value at all spacetime points, then infinitesimal transformation rule (10) becomes,

$$A'^a_\mu = A^a_\mu + f^a{}_{bc} A^b_\mu \alpha^c. \tag{11}$$

This is new compared to the $U(1)$ case when the gauge field did not transform at all in the global limit of the gauge transformation. This implies that the ***non-abelian gauge field is charged*** under the global $U(N)$ symmetry while the $U(1)$ gauge field (Maxwell) is neutral.

Since the nonabelian gauge field transforms nontrivially under the global subgroup of the local symmetry, it is natural to ask which representation (tensor) of the group does it transform like. Recalling the infinitesimal version of transformation of a field in the adjoint representation, Eq. (6), we conclude that the ***nonabelian gauge field transforms like the adjoint representation***.

3 Kinetic Term for the Gauge field

Via minimmal coupling i.e. replacing the partial derivatives of the scalar field, $\partial_\mu \mathbf{\Phi}$ by $D_\mu \mathbf{\Phi}$, we have successfully coupled the multicomponent complex scalar field to a matrix-valued vector field, A_μ, i.e. the gauge field.

$$D_\mu \mathbf{\Phi}^\dagger D^\mu \mathbf{\Phi} = \partial_\mu \mathbf{\Phi}^\dagger \partial^\mu \mathbf{\Phi} - i\, g \left(\mathbf{\Phi}^\dagger A_\mu{}^\dagger \partial^\mu \mathbf{\Phi} - \partial_\mu \mathbf{\Phi}^\dagger A^\mu \mathbf{\Phi} \right) + g^2 \mathbf{\Phi}^\dagger A^\dagger_\mu A^\mu \mathbf{\Phi}. \tag{12}$$

Clearly, the first term represents the kinetic energy term for the scalars while the two other terms represent the scalar coupling to the gauge field. However, we are yet to write down a kinetic energy term for this gauge field itself. Recall the case of the $U(1)$ gauge field, A_μ again. The naive kinetic term for the $U(1)$ gauge field, namely, $(\partial_\mu A_\nu)(\partial^\mu A^\nu)$ does not work because it is not invariant under $U(1)$ gauge (local) symmetry. Instead, the proper kinetic term was given by,

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

which was arrived upon by noticing that one might consider a term which is quadratic in derivatives of the gauge fields, as well as transforming like a tensor under $U(1)$ gauge transformation, namely,

$$e\, F_{\mu\nu}\, \phi \equiv i\, [D_\mu, D_\nu]\, \phi \tag{13}$$

It is evident that the right hand side transforms covariantly under a gauge transformation and it is quadratic in the derivative of the gauge fields as well. Then one might suggest that a similar construction be carried out for the nonabelian gauge field. Indeed working out the steps one finds

$$i\, [D_\mu, D_\nu]\, \mathbf{\Phi} \;\; = \;\; T_a\, F^a_{\mu\nu} \mathbf{\Phi}, \tag{14}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a{}_{bc} A^b_\mu A^c_\nu \tag{15}$$

is the a-th component of the *field strength tensor*, $\boldsymbol{F}_{\mu\nu} = T_a\, F^a_{\mu\nu}$. In index-free notation,

$$\boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu - i\, [\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]\, . \tag{16}$$

From the lhs of (14), it is evident that this field strength tensor should transform as a tensor under a local $U(N)$ transformation. But it is instructive to check the transformation properties of the rhs anyway. It can be checked that,

$$\boldsymbol{F}'_{\mu\nu} \;\; = \;\; U \boldsymbol{F}_{\mu\nu} U^{-1}. \tag{17}$$

Now it is easy to write down a gauge invariant kinetic term for the gauge field in terms of the field strength tensor similar to the Maxwell case,

$$\mathcal{L}_{kin}[\boldsymbol{A}_\mu] = K\, \mathrm{Tr}\, (\boldsymbol{F}_{\mu\nu} \boldsymbol{F}^{\mu\nu}) \tag{18}$$

where K is a proportionality constant. The trace is crucial for the term to be gauge invariant as can be seen explicitly,

$$\mathrm{Tr}\left(\boldsymbol{F}'_{\mu\nu} \boldsymbol{F}'^{\mu\nu}\right) = \mathrm{Tr}\left(U\, \boldsymbol{F}_{\mu\nu} \boldsymbol{F}^{\mu\nu}\, U^{-1}\right) = \mathrm{Tr}\left(\boldsymbol{F}_{\mu\nu} \boldsymbol{F}^{\mu\nu}\, U^{-1}\, U\right) = \mathrm{Tr}\, (\boldsymbol{F}_{\mu\nu} \boldsymbol{F}^{\mu\nu})\, . \tag{19}$$

We can simplify the kinetic term for the gauge field a little bit more by writing it in component notation,

$$\mathcal{L}_{kin}\left[A^a_\mu\right] = K\, \mathrm{Tr}\, (\boldsymbol{F}_{\mu\nu} \boldsymbol{F}^{\mu\nu}) = K\, \mathrm{Tr}\left(F^b_{\mu\nu} T_b\, F^{c\mu\nu} T_c\right) = K\, \mathrm{Tr}\, (T_b\, T_c)\, F^b_{\mu\nu} F^{c\mu\nu}.$$

Since T_a's can be any arbitrary representation, the value of the coefficient $\mathrm{Tr}\,(T_b\,T_c)$ is different in each representation. This is a bit of an issue because then the kinetic term normalization becomes dependent on the particular representation chosen. One gets around this by choosing the constant K be also different for each representation that the value of the product is same i.e. independent of the representation. This product is given by the convention,

$$K^{\mathrm{Adj}} = -\frac{1}{2},\ \mathrm{Tr}\left(T_a^{\mathrm{Adj}} T_b^{\mathrm{Adj}}\right) = g_{ab}$$

Thus we have the representation independent kinetic term for the nonabelian gauge field to be,

$$\mathcal{L}_{\mathrm{kin}}\left[A_\mu^a\right] = -\frac{1}{2}\, g_{ab}\, F_{\mu\nu}^a F^{b\mu\nu} \tag{20}$$

The coefficient g_{ab} plays the role of the metric in the space of the generators (Lie algebra) and is known as the *Cartan-Killing metric*.

Homework Exercises

1. Check that the proposed adjoint representation generators defined by eq.(5) indeed satisfies the necessary conditions for the Lie agebra, namely Eq.s (2) and (3).

2 The Lie algebra $su(2)$ is commonly cited as

$$[J_a, J_b] = i\,\epsilon_{abc}\,J_c.$$

where the indices a, b, c take values $1, 2, 3$. From this we can easily identify the structure constants to be,

$$f^c{}_{ab} = \epsilon_{abc}.$$

You are already familiar with the fundamental representation wherein, $J_a = \frac{\sigma_a}{2}$, where the σ_a's are the Pauli matrices. Now using (6) to construct the $su(2)$ generators (matrices) in the adjoint representation.

3. Check that the nonabelian field strength tensor defined by (14) transforms in the adjoint representation. Hint: Begin with the index free version of the field strength (16) and first prove eq.(17) using the transformation law for the gauge field (9) and then express the infinitesimal version of (17) in component form.