University of Connecticut
Department of Mathematics

MATH 2210 | # CHAPTER 2 WORKSHEET | SPRING 2019

Due: Thursday, March 7

Directions: This worksheet covers Chapter 1 from Lay's Linear Algebra book. Turn in the red-squared problems below (problems 7, 9, 12, 13, 14, and 15) on separate paper. Use only one side of the paper, remove all fringes, and staple this cover sheet to your submission.

Any of these problems is fair game for the exam (not just the red ones).
Ask questions if you're unsure about anything.

GROUP MEMBER(S) (up to four)

NAME: ____________________ NAME: ____________________
NAME: ____________________ NAME: ____________________

Section (of the student I will hand this back to):______________

Checklist for formatting:
- ☐ write every group member's name
- ☐ remove all fringes
- ☐ use only one side of the paper
- ☐ label each problem clearly
- ☐ give a clear justification for your answer
- ☐ circle your answer
- ☐ staple this coversheet to your submission

CHAPTER 2

1. Assume that the matrices mentioned in the statements below have appropriate sizes. Mark each statement True or False. Justify each answer.
   a. If A and B are $m \times n$, then both AB^T and A^TB are defined.
   b. If $AB = C$ and C has 2 columns, then A has 2 columns.
   c. Left-multiplying a matrix B by a diagonal matrix A, with nonzero entries on the diagonal, scales the rows of B.
   d. If $BC = BD$, then $C = D$.
   e. If $AC = 0$, then either $A = 0$ or $C = 0$.
   f. If A and B are $n \times n$, then $(A + B)(A - B) = A^2 - B^2$.
   g. An elementary $n \times n$ matrix has either n or $n + 1$ nonzero entries.
   h. The transpose of an elementary matrix is an elementary matrix.
   i. An elementary matrix must be square.
   j. Every square matrix is a product of elementary matrices.
   k. If A is a 3×3 matrix with three pivot positions, there exist elementary matrices $E_1, \ldots, E_p$ such that $E_p \cdots E_1 A = I$.
   l. If $AB = I$, then A is invertible.
   m. If A and B are square and invertible, then AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$.
   n. If $AB = BA$ and if A is invertible, then $A^{-1}B = BA^{-1}$.
   o. If A is invertible and if $r \neq 0$, then $(rA)^{-1} = rA^{-1}$.
   p. If A is a 3×3 matrix and the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution, then A is invertible.

2. Find the matrix C whose inverse is $C^{-1} = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Show that $A^3 = 0$. Use matrix algebra to compute the product $(I - A)(I + A + A^2)$.

4. Suppose $A^n = 0$ for some $n > 1$. Find an inverse for $I - A$.

5. Suppose an $n \times n$ matrix A satisfies the equation $A^2 - 2A + I = 0$. Show that $A^3 = 3A - 2I$ and $A^4 = 4A - 3I$.

6. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. These are *Pauli spin matrices* used in the study of electron spin in quantum mechanics. Show that $A^2 = I$, $B^2 = I$, and $AB = -BA$. Matrices such that $AB = -BA$ are said to *anticommute*.

7. Let $A = \begin{bmatrix} 1 & 3 & 8 \\ 2 & 4 & 11 \\ 1 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ 1 & 5 \\ 3 & 4 \end{bmatrix}$. Compute $A^{-1}B$ without computing A^{-1}. [*Hint:* $A^{-1}B$ is the solution of the equation $AX = B$.]

8. Find a matrix A such that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ into $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, respectively. [*Hint:* Write a matrix equation involving A, and solve for A.]

9. Suppose $AB = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$. Find A.

10. Suppose A is invertible. Explain why A^TA is also invertible. Then show that $A^{-1} = (A^TA)^{-1}A^T$.

11. Let $x_1, \ldots, x_n$ be fixed numbers. The matrix below, called a *Vandermonde matrix*, occurs in applications such as signal processing, error-correcting codes, and polynomial interpolation.

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

Given $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, suppose $\mathbf{c} = (c_0, \ldots, c_{n-1})$ in $\mathbb{R}^n$ satisfies $V\mathbf{c} = \mathbf{y}$, and define the polynomial

$p(t) = c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1}$.

   a. Show that $p(x_1) = y_1, \ldots, p(x_n) = y_n$. We call $p(t)$ an *interpolating polynomial for the points* $(x_1, y_1), \ldots, (x_n, y_n)$ because the graph of $p(t)$ passes through the points.
   b. Suppose $x_1, \ldots, x_n$ are distinct numbers. Show that the columns of V are linearly independent. [*Hint:* How many zeros can a polynomial of degree $n - 1$ have?]
   c. Prove: "If $x_1, \ldots, x_n$ are distinct numbers, and $y_1, \ldots, y_n$ are arbitrary numbers, then there is an interpolating polynomial of degree $\leq n - 1$ for $(x_1, y_1), \ldots, (x_n, y_n)$."

12. Let $A = LU$, where L is an invertible lower triangular matrix and U is upper triangular. Explain why the first column

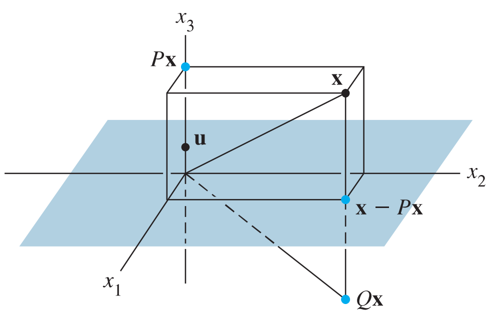
of A is a multiple of the first column of L. How is the second column of A related to the columns of L?

13. Given $\mathbf{u}$ in $\mathbb{R}^n$ with $\mathbf{u}^T\mathbf{u} = 1$, let $P = \mathbf{u}\mathbf{u}^T$ (an outer product) and $Q = I - 2P$. Justify statements (a), (b), and (c).

a. $P^2 = P$ b. $P^T = P$ c. $Q^2 = I$

The transformation $\mathbf{x} \mapsto P\mathbf{x}$ is called a *projection*, and $\mathbf{x} \mapsto Q\mathbf{x}$ is called a *Householder reflection*. Such reflections are used in computer programs to create multiple zeros in a vector (usually a column of a matrix).

14. Let $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$. Determine P and Q as in Exercise 13, and compute $P\mathbf{x}$ and $Q\mathbf{x}$. The figure shows that $Q\mathbf{x}$ is the reflection of $\mathbf{x}$ through the x_1x_2-plane.



A Householder reflection through the plane $x_3 = 0$.

15. Suppose $C = E_3E_2E_1B$, where E_1, E_2, and E_3 are elementary matrices. Explain why C is row equivalent to B.

16. Let A be an $n \times n$ singular matrix. Describe how to construct an $n \times n$ nonzero matrix B such that $AB = 0$.

17. Let A be a 6×4 matrix and B a 4×6 matrix. Show that the 6×6 matrix AB cannot be invertible.

18. Suppose A is a 5×3 matrix and there exists a 3×5 matrix C such that $CA = I_3$. Suppose further that for some given $\mathbf{b}$ in $\mathbb{R}^5$, the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution. Show that this solution is unique.