3.1 # 4, 8, 11, 13, 20, 21, 31, 32, 37, 39

4.) Compute the determinant using a cofactor expansion across the first row. Also compute it by a cofactor expansion down the second column.

1st row
$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix} = (-1)^{1+1}(1)\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} + (-1)^{1+2}(3)\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + (-1)^{1+3}(5)\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = -2-3+25 = 20$$

2nd column
$$(-1)^{1+2}(3)\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + (-1)^{2+2}(1)\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} + (-1)^{3+2}(4)\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = -3+(-13)-(-36) = 20$$

8.) Compute the determinant using a cofactor expansion across the first row.

$$\begin{vmatrix} 8 & 1 & 6 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{vmatrix} = (-1)^{1+1}(8)\begin{vmatrix} 0 & 3 \\ -2 & 5 \end{vmatrix} + (-1)^{1+2}(1)\begin{vmatrix} 4 & 3 \\ 3 & 5 \end{vmatrix} + (-1)^{1+3}(6)\begin{vmatrix} 4 & 0 \\ 3 & -2 \end{vmatrix} = 48-11-48 = -11$$

11.) Compute the determinant by cofactor expansion. At each step choose a row or column that involves the least computation.

$$\begin{vmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix} = (-1)^{1+1}(3)\begin{vmatrix} -2 & 3 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{vmatrix} + (-1)^{2+1}(0) + (-1)^{3+1}(0) + (-1)^{4+1}(0)$$

$$= 3\left((-1)^{1+1}(-2)\begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix} + (-1)^{2+1}(0) + (-1)^{3+1}(0)\right) = 3(-2)\begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix} = -12$$

* could also have started w/ row 4.

13.) $\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix} = 0 + 0 + (-1)^{2+3}(2)\begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix} + 0 + 0$

$= -2\left(0 + (-1)^{2+2}(3)\begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix} + 0 + 0\right) = (-2)(3)\left((-1)^{1+1}(4)\begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} + (-1)^{2+1}(5)\begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix} + 0\right)$

$= -6(4-5) = 6$

* or start w/ column 2

20.) State the row operation and describe how it affects the determinant.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$ The row operation is kR_2.

$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ $\det\begin{bmatrix} a & b \\ kc & kd \end{bmatrix} = kad - kbc = k(ad-bc)$

Multiplying R_2 by k multiplies the determinant by k.

21.) $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 5+3k & 6+4k \end{bmatrix}$ The row operation is $kR_1 + R_2$

$\det\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = 3(6) - 4(5)$ $\det\begin{bmatrix} 3 & 4 \\ 5+3k & 6+4k \end{bmatrix} = 3(6+4k) - 4(5+3k)$

$= 3(6) + \cancel{3(4k)} - 4(5) - \cancel{4(3k)}$

The row operation $kR_1 + R_2$ has no effect on the determinant.

3.1 continued

31.) What is the determinant of an elementary row replacement matrix?

An elementary row replacement matrix has 1's in the diagonal, one other nonzero entry and the rest zeros. So it is a triangular matrix. Therefore the determinant is the product of the entries on the main diagonal which is 1.

32.) What is the determinant of an elementary scaling matrix with k on the diagonal?

An elementary scaling matrix has 1's on the main diagonal except one position and zeros everywhere else. It is a triangular matrix so its determinant is k.

37.) Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Write $5A$. Is $\det 5A = 5 \det A$?

$5A = \begin{bmatrix} 15 & 5 \\ 20 & 10 \end{bmatrix}$ $\det 5A = 50$ $\det A = 2$ No $\det 5A \neq 5 \det A$

39.) True/False (A is $n \times n$ matrix)

a.) An $n \times n$ ~~de~~determinant is defined by determinants of $(n-1)\times(n-1)$ submatrices.

b.) The (i,j)-cofactor of a matrix A is the matrix A_{ij} obtained by deleting from A its ith row and jth column.

a.) True b.) False

3.2 # 2, 3, 8, 10, 16, 17, 20, 26, 27, 32, 34, 40

2.) The equation illustrates a property of determinants. State the property.

$$\begin{vmatrix} 2 & -6 & 4 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & -3 & 2 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix}$$

If one row of A is multiplied by k to produce B then $\det B = k \cdot \det A$.

3.)
$$\begin{vmatrix} 1 & 3 & -4 \\ 2 & 0 & -3 \\ 5 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -4 \\ 0 & -6 & 5 \\ 5 & -4 & 7 \end{vmatrix}$$

If a multiple of one row of A is added to another row to produce a matrix B, then $\det A = \det B$.

8.) Find the determinant by row reduction to echelon form.

$$\begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix} \begin{matrix} \\ \\ -2R_1+R_3 \\ 3R_1+R_4 \end{matrix} \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 2 & 4 & -10 \end{bmatrix} \begin{matrix} \\ \\ R_2+R_3 \\ -2R_2+R_3 \end{matrix} \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Triangular matrix ⇒ multiply entries in main diagonal to get determinant. The determinant is $(1)(1)(0)(0) = 0$.

10.)
$$A = \begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{bmatrix} \begin{matrix} \\ \\ 2R_1+R_3 \\ -3R_1+R_4 \\ -3R_1+R_5 \end{matrix} \begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & -2 & 0 & 8 & -1 \\ 0 & -4 & 8 & 2 & 13 \end{bmatrix} \begin{matrix} \\ \\ \\ R_2+R_4 \\ 2R_2+R_5 \end{matrix} \begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & -4 & 7 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = B$$

no change in det   no change in det

So $\det A = \det B$

Interchanging rows means $\det B = -\det C$

$$\begin{bmatrix} 1 & 3 & -1 & 0 & 2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & -4 & 7 & -7 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = C$$

Since C is triangular

$\det C = 1(2)(-4)(3)(1) = -24$

$\det A = \det B = -\det C$

$= -(-24) = \boxed{24}$

16.) Find the determinant where $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$.

$$\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix} = 3(7) = 21$$

3.2

17.) $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -7$

20.) $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$

26.) Use determinants to decide if the set of vectors is linearly independent.

$$\begin{bmatrix} 3 \\ 5 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -2 & 2 & 3 \\ 0 & -1 & -6 & 5 \\ 0 & 3 & 0 & -6 \\ -3 & 0 & 7 & 4 \end{bmatrix} \sim \begin{bmatrix} -3 & 0 & 7 & 4 \\ 0 & -1 & -6 & 5 \\ 0 & 3 & 0 & -6 \\ 0 & -2 & 2 & 3 \end{bmatrix} \begin{array}{l} \\ \\ 3R_2+R_3 \\ -2R_2+R_4 \end{array}$$

$\det A = -\det B$

$$\underbrace{\begin{bmatrix} -3 & 0 & 7 & 4 \\ 0 & -1 & -6 & 5 \\ 0 & 0 & -18 & 9 \\ 0 & 0 & 14 & -7 \end{bmatrix}}_{B} \begin{array}{l} \\ \\ R_3/(-9) \\ R_4/(7) \end{array} \underbrace{\begin{bmatrix} -3 & 0 & 7 & 4 \\ 0 & -1 & -6 & 5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & -1 \end{bmatrix}}_{C} -R_3+R_4 \underbrace{\begin{bmatrix} -3 & 0 & 7 & 4 \\ 0 & -1 & -6 & 5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{D}$$

$\det D = (-3)(-1)(2)(0) = 0$

$\det B = (\frac{-1}{9})(\frac{1}{7})\det C$ $\qquad$ $\det C = \det D$

$\det A = -\det B = (\frac{1}{9})(\frac{1}{7})\det C = (\frac{1}{9})(\frac{1}{7})\det D = 0$ Since determinant of A is zero, A is not invertible which means the vectors are linearly dependent.

27.) True/False (A, B are $n \times n$ matrices)

a.) A row replacement operation does not affect the determinant of a matrix.

b.) The determinant of A is the product of pivots in any echelon form U of A, multiplied by $(-1)^r$, where r is the number of row interchanges made during row reduction from A to U.

c.) If the columns of A are linearly dependent, then $\det A = 0$.

d.) $\det(A+B) = \det A + \det B$.

a.) True $\quad$ b.) True $\quad$ c.) True $\quad$ d.) False

3.2 Continued

32.) Find a formula for $\det(rA)$ when A is an $n \times n$ matrix.

rA multiplies r to each rows of A and each time it multiplies the determinant of A by r. Since there are n rows, $\det(rA) = r^n \det A$.

34.) Let A and P be square matrices, with P invertible. Show that $\det(PAP^{-1}) = \det A$.

$$\det(PAP^{-1}) = (\det P)(\det A)(\det P^{-1}) = (\det P)(\det P^{-1})(\det A)$$
$$= (\det PP^{-1})(\det A) = (\det I)(\det A) = 1 \cdot \det A = \det A$$

40.) Let A and B be 4×4 matrices, with $\det A = -1$ and $\det B = 2$. Compute:

a.) $\det AB$ b.) $\det B^5$ c.) $\det 2A$ d.) $\det A^TA$ e.) $\det B^{-1}AB$

a.) $\det AB = (\det A)(\det B) = -2$

b.) $\det B^5 = (\det B)^5 = 32$

c.) $\det 2A = 2^4 \det A = 16(-1) = -16$

d.) $\det A^TA = (\det A^T)(\det A) = (\det A)(\det A) = 1$

e.) $\det(B^{-1}AB) = \det A = -1$

3.3 # 4, 5, 6, 22, 23, 26, 29, 30

4.) Use cramer's rule to compute the solution.

$$-5x_1 + 3x_2 = 9$$
$$3x_1 - x_2 = -5$$

$A = \begin{bmatrix} -5 & 3 \\ 3 & -1 \end{bmatrix}$ $\det A = -4$ $\vec{b} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$

$A_1(\vec{b}) = \begin{bmatrix} 9 & 3 \\ -5 & -1 \end{bmatrix}$ $\det A_1(\vec{b}) = 6$ $A_2(\vec{b}) = \begin{bmatrix} -5 & 9 \\ 3 & -5 \end{bmatrix}$ $\det A_2(\vec{b}) = -2$ $\vec{x} = \begin{bmatrix} -6/4 \\ 2/4 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$

5.)

$$2x_1 + x_2 = 7$$
$$-3x_1 + x_3 = -8$$
$$x_2 + 2x_3 = -3$$

$A = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ $\det A = 2(-1) - 1(-6) + 0 = 4$

$A_1(\vec{b}) = \begin{bmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{bmatrix}$ $\det A_1(\vec{b}) = 7(-1) - 1(-13) + 0 = 6$

$A_2(\vec{b}) = \begin{bmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{bmatrix}$ $\det A_2(\vec{b}) = 2(-13) + 3(14) + 0 = 16$

$\vec{x} = \begin{bmatrix} 6/4 \\ 16/4 \\ -14/4 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 4 \\ -7/2 \end{bmatrix}$

$A_3(\vec{b}) = \begin{bmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{bmatrix}$ $\det A_3(\vec{b}) = 2(8) + 3(-10) + 0 = -14$

6.)

$$2x_1 + x_2 + x_3 = 4$$
$$-x_1 + \quad +2x_3 = 2$$
$$3x_1 + x_2 + 3x_3 = -2$$

$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix}$ $\det A = 1(-9) - 0 + 1(5) = -4$

$A_1(\vec{b}) = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{bmatrix}$ $\det A_1(\vec{b}) = 1(10) - 0 + 1(6) = 16$

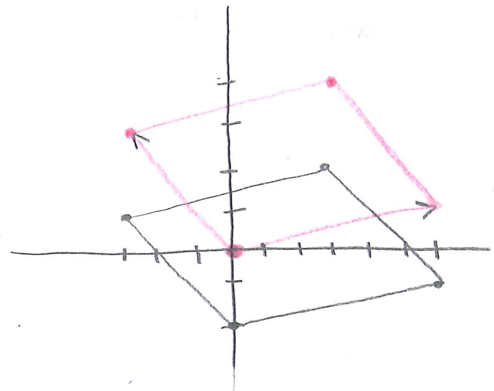
$A_2(\vec{b}) = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{bmatrix}$ $\det A_2(\vec{b}) = 2(10) + 1(14) + 3(6) = 52$ $\vec{x} = \begin{bmatrix} 16/-4 \\ 52/-4 \\ 4/-4 \end{bmatrix} = \begin{bmatrix} -4 \\ -13 \\ -1 \end{bmatrix}$

$A_3(\vec{b}) = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{bmatrix}$ $\det A_3(\vec{b}) = 1(-4) - 0 + 1(8) = 4$

22.) Find the area of the parallelogram whose vertices are $(0,-2), (6,-1), (-3,1), (3,2)$

$(0,0), (6,1), (-3,3), (3,4)$

$A = \begin{bmatrix} -3 & 6 \\ 3 & 1 \end{bmatrix}$ $\quad |\det A| = |-21| = 21$



The area of the parallelogram is 21

23.) Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1,0,-2), (1,2,4)$ and $(7,1,0)$.

$A = \begin{bmatrix} 1 & 1 & 7 \\ 0 & 2 & 1 \\ -2 & 4 & 0 \end{bmatrix}$ $\quad |\det A| = |1(-4) - 0 + (-2)(-13)| = 22$

26.) Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation, and let $\vec{p}$ be a vector and S a set in $\mathbb{R}^m$. Show that the image of $\vec{p}+S$ under T is the translated set $T(\vec{p}) + T(S)$ in $\mathbb{R}^n$.

Let $\vec{v}$ be in S. Then $\vec{p}+\vec{v}$ is in $\vec{p}+S$ and $T(\vec{p}+\vec{v}) = T(\vec{p}) + T(\vec{v})$ which is in $T(\vec{p}) + T(S)$. For the other direction any vector in $T(\vec{p})+T(S)$ is of the form $T(\vec{p})+T(\vec{v})$ for some v in S. $T(\vec{p})+T(\vec{v}) = T(\vec{p}+\vec{v})$ So this vector is the image of something in $\vec{p}+S$ under T.

3.3 continued

29.) Find a formula for the area of the triangle whose vertices are $\vec{0}$, $\vec{v}_1$, and $\vec{v}_2$ in $\mathbb{R}^2$.

$\frac{1}{2}\det[\vec{v}_1, \vec{v}_2]$

30.) Let R be the triangle w/ vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3). Show that $\{\text{area of triangle}\} = \frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$

$(0,0), (x_2-x_1, y_2-y_1), (x_3-x_1, y_3-y_1)$

$$\text{area} = \frac{1}{2}\det\begin{bmatrix} x_2-x_1 & x_3-x_1 \\ y_2-y_1 & y_3-y_1 \end{bmatrix} = \frac{1}{2}\det\begin{bmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{bmatrix}$$

since $\det A = \det A^T$ when A is $n \times n$

$$= \frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2-x_1 & y_2-y_1 & 0 \\ x_3-x_1 & y_3-y_1 & 0 \end{bmatrix} = \frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

row operations R_1+R_2
R_1+R_3

4.1 # 1, 3, 8, 12, 13, 15, 17, 22, 23, 31, 32

1.) Let V be the first quadrant in the xy-plane; that is, let

$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$

a.) If $\vec{u}$ and $\vec{v}$ are in V, is $\vec{u} + \vec{v}$ in V? Why

b.) Find a specific vector $\vec{u}$ in V and a specific scalar c such that $c\vec{u}$ is not in V. (This is enough to show V is not a vector space)

a.) Yes. The entries in $\vec{u}$ and $\vec{v}$ are non negative, so their sum is non negative too which means $\vec{u} + \vec{v}$ is in V.

b.) Many answers, but if $c = -1$ and $\vec{u}$ is in V, then $c\vec{u}$ is not in V.

3.) Let H be the set of points inside and on the unit circle in the xy-plane. That is, let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}$. Find a specifice example - two vectors or a vector & a scalar - to show that H is not a subspace of $\mathbb{R}^2$.

Many answers. If $\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\vec{u}$ is in H, but $4\vec{u} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ is not in H. Therefore H is not closed under scalar multiplication and is not a subspace of $\mathbb{R}^2$.

8.) Determine if the set of all polynomials in $\mathbb{P}_n$ such that $\vec{p}(0) = 0$ is a subspace of $\mathbb{P}_n$.

Call our set H. The zero polynomial is in H. If $\vec{p} \in H$, then for any c $c\vec{p}(0) = c(0) = 0$, so $c\vec{p}(0) \in H$. If $\vec{p}$ and $\vec{q}$ are in H, then $(\vec{p} + \vec{q})(0) = \vec{p}(0) + \vec{q}(0) = 0 + 0 = 0$, so $\vec{p} + \vec{q} \in H$. Thus H is a subspace.

4.1

12.) Let W be the set of all vectors of the form $\begin{bmatrix} 2s+4t \\ 2s \\ 2s-3t \\ 5t \end{bmatrix}$.
Show that W is a subspace of $\mathbb{R}^4$.

If $\vec{u} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 5 \end{bmatrix}$, then $W = \text{Span}(\vec{u}, \vec{v})$.

(s and t are the coefficients in the linear combination of $\vec{u}$ & $\vec{v}$)
Since $\vec{u}, \vec{v}$ are in $\mathbb{R}^4$, Span$(\vec{u}, \vec{v})$ is a subspace of $\mathbb{R}^4$. Thus W is a subspace of $\mathbb{R}^4$.

13.) Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

a.) Is $\vec{w}$ in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? How many vectors are in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?
b.) How many vectors are in Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?
c.) Is $\vec{w}$ in the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? Why?

a.) No, 3

b.) Infinitely many

c.) In other words, is $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \mid \vec{w}]$ consistent?

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right] \underset{\substack{\text{many row} \\ \text{operations} \\ \text{later}}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Yes, $\vec{w}$ is in the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

15.) Let W be the set of all vectors of the form $\begin{bmatrix} 2a+3b \\ -1 \\ 2a-5b \end{bmatrix}$, where a, b are arbitrary real numbers. Find a set S of vectors that spans W or give an example to show W is not a vector space.

The zero vector is not in W, so W is not a vector space.

4.1 continued

17.) (Same directions as #15)

$$\begin{bmatrix} 2a-b \\ 3b-c \\ 3c-a \\ 3b \end{bmatrix} \quad \vec{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} \qquad W = \text{span}\{\vec{u}, \vec{v}, \vec{w}\}$$

$$S = \{\vec{u}, \vec{v}, \vec{w}\}$$

22.) Let F be a fixed 3×2 matrix, and let H be the set of all matrices A in $M_{2\times4}$ with the property that $FA = 0$. Determine if H is a subspace of $M_{2\times4}$.

The zero matrix is in H. If A, B are in H then $F(A+B) = FA + FB = 0$. so $F(A+B) \in H$. For a scalar c, $F(cA) = c(FA) = c(0) = 0$.

Therefore H is a subspace of $M_{2\times4}$

23.) True/False

a.) If $\vec{f}$ is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and if $\vec{f}(t) = 0$ for some t, then $\vec{f}$ is the zero vector in V.

b.) A vector is an arrow in three-dimensional space.

c.) A subset H of a vector space V is a subspace of V if the zero vector is in H.

d.) A subspace is also a vector space.

a.) False b.) False c.) False d.) True

31.) Let $\vec{u}$ and $\vec{v}$ be vectors in a vector space V, and let H be any subspace of V that contains both $\vec{u}$ & $\vec{v}$. Explain why H also contains Span$\{\vec{u}, \vec{v}\}$. This shows that Span$\{\vec{u}, \vec{v}\}$ is the smallest subspace of V that contains both $\vec{u}$ & $\vec{v}$.

A subspace containing $\vec{u}$ & $\vec{v}$ must contain all linear combinations of $\vec{u}$ and $\vec{v}$ and thus contains Span$\{\vec{u}, \vec{v}\}$.

32.) Let H and K be subspaces of a vector space V. The intersection of H and K, written $H \cap K$, is the set of $\vec{v} \in V$ that belong to both H and K. Show that $H \cap K$ is a subspace of V. Then give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace.

Since H & K are vector spaces, $\vec{0} \in H$, $\vec{0} \in K$, so $\vec{0} \in H \cap K$.
Suppose $\vec{u}, \vec{v} \in H \cap K$. Then for a scalar c, $c\vec{u} \in H$ and $c\vec{u} \in K$ so $c\vec{u} \in H \cap K$. Also, since $\vec{u}, \vec{v} \in H \cap K$, $\vec{u}, \vec{v} \in H$ and $\vec{u}, \vec{v} \in K$, therefore $\vec{u} + \vec{v} \in H$ and $\vec{u} + \vec{v} \in K$ since H, K are both vector spaces. Thus $\vec{u} + \vec{v} \in H \cap K$. Hence $H \cap K$ is a vector space.

In $\mathbb{R}^2$, let H be the x-axis, K be the y-axis. Then H & K are subspaces, but their union is not because it is not closed under addition. For example $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in H$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in K$, but $\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in $H \cup K$. So in general, the union of two vector spaces is not a vector space.

4.2 # 3, 6, 11, 14, 17, 19, 21, 24, 25, 33, 34

3) $A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$ Find an explicit description of Nul A by listing vectors that span the null space.

Solve $A\vec{x} = \vec{0}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 4 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{array}\right]$$

x_3, x_4 free

$x_1 = 2x_3 - 4x_4$

$x_2 = -3x_3 + 2x_4$

$$\vec{x} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

The spanning set of Nul A is

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

6) $A = \begin{bmatrix} 1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Solve $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-3R_2+1} \begin{bmatrix} 1 & 0 & 5 & -6 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3, x_4, x_5 free

$x_1 = -5x_3 + 6x_4 - x_5$

$x_2 = 3x_3 - x_4$

$$\vec{x} = x_3 \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The spanning set of Nul A is

$$\left\{ \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

11.) Either use an appropriate theorem to show that the given set W is a vector space, or find a specific example to the contrary.

$$\left\{ \begin{bmatrix} s-2t \\ 3+3s \\ 3s+t \\ 2s \end{bmatrix} : s,t \text{ real} \right\}$$

If W were a vector space, it would contain the zero vector, but the second entry is zero when $s=-1$ and the last entry is zero when $s=0$. Therefore this set doesn't contain the zero vector. W is not a vector space.

14.) (same directions as # 11)

$$\left\{\begin{bmatrix} -s+3t \\ s-2t \\ 5s-t \end{bmatrix} : s,t \text{ real}\right\}$$

$$\vec{w} = s\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

Since any $\vec{w}$ in W can be written this way,

$W = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}\right\} = \text{Col}\begin{bmatrix} -1 & 3 \\ 1 & -2 \\ 5 & -1 \end{bmatrix}$. Since the column space of a matrix is a subspace, W is a vector space.

17.) a.) Find k such that Nul A is a subspace of $\mathbb{R}^k$
b.) Find k such that Col A is a subspace of $\mathbb{R}^k$

$$A = \begin{bmatrix} 6 & -4 \\ -3 & 2 \\ -9 & 6 \\ 9 & -6 \end{bmatrix}$$

A is 4×2 so Nul A is a subspace of $\mathbb{R}^2$ $(k=2)$ and Col A is a subspace of $\mathbb{R}^4$ $(k=4)$.

19.) $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$

A is 2×5 so Nul A is a subspase of $\mathbb{R}^5$ and Col A is a subspace of $\mathbb{R}^2$.

21.) With A as in #17, find a non-zero vector in Nul A and a nonzero vector in Col A.

$\begin{bmatrix} 6 \\ -3 \\ -9 \\ 9 \end{bmatrix}$ is a non-zero vector in Col A since Col $A = \text{Span}\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n\}$.

Solve $A\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 6 & -4 & 0 \\ -3 & 2 & 0 \\ -9 & 6 & 0 \\ 9 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2/3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 is free

$x_1 = 2/3\, x_2$.

$$\vec{x} = x_2\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

All we have to do to find a non-zero vector in Nul A is choose a nonzero value for x_2 and find $\vec{x}$.

For example if $x_2 = 3$ $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

4.2 Continued

24.) $A = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$ Determine if $\vec{w}$ is in Col A. Is $\vec{w}$ in Nul A?

$$\left[\begin{array}{cccc|c} 10 & -8 & -2 & -2 & 2 \\ 0 & 2 & 2 & -2 & 2 \\ 1 & -1 & 6 & 0 & 0 \\ 1 & 1 & 0 & -2 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since $A\vec{x} = \vec{w}$ is consistent, $\vec{w}$ is in Col A.

$$\begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 20-16+0-4 \\ 0+4+0-4 \\ 2-2+0+0 \\ 2+2+0-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $A\vec{w} = \vec{0}$, $\vec{w}$ is in Nul A.

25.) True/False A is an $m \times n$ matrix.

a.) The null space of A is the soln set of $A\vec{x} = \vec{0}$. TRUE

b.) The null space of an $m \times n$ matrix is in $\mathbb{R}^m$. FALSE

c.) The column space of A is the range of the mapping $\vec{x} \mapsto A\vec{x}$. TRUE

d.) If the equation $A\vec{x} = \vec{b}$ is consistent, then Col A is $\mathbb{R}^m$. FALSE

e.) The Kernel of a linear transformation is a vector space. True

f.) Col A is the set of all vectors that can be written as $A\vec{x}$ for some $\vec{x}$. TRUE

4.2

33.) Let $M_{2\times 2}$ be the vector space of all 2×2 matrices and define $T: M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^T$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

a.) Show that T is a linear transformation.

$$T(A+B) = (A+B) + (A+B)^T = A + B + A^T + B^T = (A + A^T) + (B + B)^T$$
$$= T(A) + T(B)$$

$$T(cA) = cA + (cA)^T = cA + c(A^T) = c(A + A^T) = cT(A).$$

b.) Let B be any element of $M_{2\times 2}$ such that $B^T = B$. Find an A in $M_{2\times 2}$ such that $T(A) = B$.

If $T(A) = B$ then $A + A^T = B$. Suppose $A = \frac{1}{2}B$, then $A^T = (\frac{1}{2}B)^T = \frac{1}{2}B^T$

and $T(A) = A + A^T = \frac{1}{2}B + \frac{1}{2}B^T = \frac{1}{2}B + \frac{1}{2}B = B.$

c.) Show that the range of T is the set of B in $M_{2\times 2}$ with the Property that $B^T = B$.

In part (b) we showed if $B = B^T$, then B is in the range of T.

Now we show the other direction ie. If B is in the range of T then $B = B^T$.

Suppose $B = A + A^T$ then $B^T = (A + A^T)^T = A^T + A = B.$

d.) Describe the Kernel of T.

The Kernel is the set of all A such that $T(A) = 0$.

$$A + A^T = 0 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow a, d = 0 \text{ and } b = -c$$

Kernel of T is:

$$\left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \text{ is real} \right\}$$

4.2 continued

34) Define $T: C[0,1] \to C[0,1]$ as follows: For $\vec{f} \in C[0,1]$, let $T(\vec{f})$ be the antiderivative $\vec{F}$ of $\vec{f}$ such that $\vec{F}(0)=0$. Show that T is a linear transformation and describe the Kernel of T.

Let $\vec{f}, \vec{g}$ be elements in $C[0,1]$.

$T(\vec{f}+\vec{g})$ is the antiderivative of $\vec{f}+\vec{g}$, from calculus we know this is the antiderivative of $\vec{f}$ plus the antiderivative of $\vec{g}$.

So $T(\vec{f}+\vec{g}) = \vec{F}+\vec{G}$ such that $(\vec{F}+\vec{G})(0)=0$.

Then $T(\vec{f}+\vec{g}) = T(\vec{f})+T(\vec{g})$. Similarly,

$T(c\vec{f}) = cT(\vec{f})$.

The Kernel of T is the set of all functions $\vec{f}$ whose antiderivative is zero and $\vec{F}(0)=0$. Therefore $\vec{f}=\vec{0}$. The Kernel of T is $\{\vec{0}\}$.

4.3 # 3, 4, 8, 10, 14, 15, 21, 23, 24, 29, 30, 31

3.) Determine whether the sets are bases for $\mathbb{R}^3$. Of the sets that are not bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$.

$$\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}3\\1\\-4\end{bmatrix}, \begin{bmatrix}-2\\-1\\1\end{bmatrix} \qquad \begin{bmatrix}1&3&-2\\0&1&-1\\-3&-4&1\end{bmatrix} \sim \begin{bmatrix}1&0&1\\0&1&-1\\0&0&0\end{bmatrix}$$

Since we have an $n\times n$ matrix without n pivots, by the IMT, the set is linearly dependent and do not span $\mathbb{R}^3$. The set is not a basis for $\mathbb{R}^3$.

4.)

$$\begin{bmatrix}2\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\end{bmatrix}, \begin{bmatrix}-8\\5\\4\end{bmatrix} \qquad \begin{bmatrix}2&2&-8\\-1&-3&5\\1&2&4\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

We have an $n\times n$ matrix with n pivots, so by IMT, the columns are lin. indep and span $\mathbb{R}^3$. The set is a basis for $\mathbb{R}^3$.

8.)

$$\begin{bmatrix}1\\-2\\3\end{bmatrix}, \begin{bmatrix}0\\3\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\5\end{bmatrix}, \begin{bmatrix}0\\0\\-1\end{bmatrix} \qquad \begin{bmatrix}1&0&2&0\\-2&3&-1&0\\3&-1&5&-1\end{bmatrix} \sim \begin{bmatrix}1&0&2&0\\0&1&1&0\\0&0&0&1\end{bmatrix}$$

(IMT not applicable) Since there is a pivot in every row the columns of A span $\mathbb{R}^3$. Since there is a free variable, the columns are not lin. indep. So the set is not a basis for $\mathbb{R}^3$.

10.) Find a basis for the null space of the matrix.

$$\begin{bmatrix}1&1&-2&1&5\\0&1&0&-1&-2\\0&0&-8&0&16\end{bmatrix} \sim \begin{bmatrix}1&0&0&2&-9\\0&1&0&-1&10\\0&0&1&0&-2\end{bmatrix}$$

x_4, x_5 free

$x_1 = -2x_4 + 9x_5$

$x_2 = x_4 - 10x_5$

$x_3 = 2x_5$

$$\vec{x} = x_4\begin{bmatrix}-2\\1\\0\\1\\0\end{bmatrix} + x_5\begin{bmatrix}9\\-10\\2\\0\\1\end{bmatrix}$$

Basis for the null space:

$$\left\{\begin{bmatrix}-2\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}9\\-10\\2\\0\\1\end{bmatrix}\right\}$$

14.) Assume A is row equivalent to B. Find bases for Nul A and Col A.

$$A = \begin{bmatrix}1&2&3&-4&8\\1&2&0&2&8\\2&4&-3&10&9\\3&6&0&6&9\end{bmatrix}$$

$$B = \begin{bmatrix}1&2&0&2&5\\0&0&3&-6&3\\0&0&0&0&-7\\0&0&0&0&0\end{bmatrix} \sim \begin{bmatrix}1&2&0&2&0\\0&0&1&-2&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}$$

echelon form

x_2, x_4 free

$x_1 = -2x_2 - 2x_4$

$x_3 = 2x_4$

$x_5 = 0$

$$\vec{x} = x_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-2\\0\\2\\1\\0\end{bmatrix}$$

Basis for Nul A is:

$$\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\2\\1\\0\end{bmatrix}\right\}$$

Basis for Col A: We know from B the pivot columns are the 1st, 3rd, 5th columns. So looking at A we get the Basis for Col A:

$$\left\{\begin{bmatrix}1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\0\\-3\\0\end{bmatrix}, \begin{bmatrix}8\\8\\9\\9\end{bmatrix}\right\}$$

15.) Find a basis for the space spanned by the given vectors.

$\begin{bmatrix}1\\0\\-2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\-2\\-8\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\10\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\-6\\9\end{bmatrix}$

We want to find a basis for Col A.

$$A = \begin{bmatrix}1 & 0 & 2 & 2 & 3\\ 0 & 1 & -2 & -1 & -1\\ -2 & 2 & -8 & 10 & -6\\ 3 & 3 & 0 & 3 & 9\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2 & 0 & 0\\ 0 & 1 & -2 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix}1\\0\\-2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\-1\\10\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\-6\\9\end{bmatrix} \right\}$

21.) True / False

a.) A single vector by itself is linearly dependent.

b.) If $H = \text{Span}\{\vec{b}_1, \dots \vec{b}_p\}$, then $\{\vec{b}_1, \dots \vec{b}_p\}$ is a basis for H.

c.) The columns of an invertible nxn matrix form a basis for $\mathbb{R}^n$.

d.) A basis is a spanning set that is as large as possible.

e.) In some cases, the linear dependence relations among the columns of a matrix can be affected by certain elementary row operations on the matrix.

a.) False b.) False c.) True d.) False e.) False

23.) Suppose $\mathbb{R}^4 = \text{Span}\{\vec{v}_1, \dots \vec{v}_4\}$. Explain why $\{\vec{v}_1, \dots \vec{v}_4\}$ is a basis for $\mathbb{R}^4$.

Let $A = [\vec{v}_1 \dots \vec{v}_4]$. Then A is a 4×4 matrix. Since its columns span $\mathbb{R}^4$, the columns of A are linearly independent by the IMT. Since $\{\vec{v}_1, \dots \vec{v}_4\}$ is linearly independent and spans $\mathbb{R}^4$, it is a basis for $\mathbb{R}^4$.

24.) Let $\mathcal{B} = \{\vec{v}_1, \dots \vec{v}_n\}$ be a linearly independent set in $\mathbb{R}^n$. Explain why $\mathcal{B}$ must be a basis for $\mathbb{R}^n$.

Let $B = [\vec{v}_1 \dots \vec{v}_n]$. B is an nxn matrix whose columns are linearly independent. By the IMT, the columns of B span $\mathbb{R}^n$. Therefore $\mathcal{B}$ is linearly independent, spans $\mathbb{R}^n$ and hence $\mathcal{B}$ is a basis for $\mathbb{R}^n$.

4.3 continued

29.) Let $S = \{\vec{v}_1, \dots \vec{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$, with $k<n$. Use a theorem from Section 1.4 to explain why S cannot be a basis for $\mathbb{R}^n$.

Let $A = [\vec{v}_1 \dots \vec{v}_k]$. Since A has more rows than columns, there can't be a pivot in every row. Therefore by thm 4 in 1.4 the columns of A do not span $\mathbb{R}^n$. Thus S is not a basis for $\mathbb{R}^n$.

30.) Let $S = \{\vec{v}_1, \dots \vec{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$, with $k>n$. Use a theorem from chapter 1 to explain why S cannot be a basis for $\mathbb{R}^n$.

Since S has more vectors than entries, S is linearly dependent by theorem 8 in 1.7. Therefore S cannot be a basis for $\mathbb{R}^n$.

31.) Let V, W be vector spaces, $T: V \to W$ be a linear transformation and $\{\vec{v}_1, \dots \vec{v}_p\}$ is a subset of V. Show that if $\{\vec{v}_1, \dots \vec{v}_p\}$ is linearly dependent in V, then the set of images, $\{T(\vec{v}_1), \dots T(\vec{v}_p)\}$ is linearly dependent in W.

Since $\{\vec{v}_1, \dots \vec{v}_p\}$ is linearly dependent, there exist scalars $c_1, \dots c_{p-1}$ such that $\vec{v}_p = c_1\vec{v}_1 + \dots + c_{p-1}\vec{v}_{p-1}$. Applying the linear trans. T to this gives us $T(\vec{v}_p) = T(c_1\vec{v}_1 + \dots + c_{p-1}\vec{v}_{p-1}) = c_1T(\vec{v}_1) + \dots + c_{p-1}T(\vec{v}_{p-1})$ therefore $\{T(\vec{v}_1), \dots T(\vec{v}_p)\}$ is linearly dependent.

(This also shows if $\{T(\vec{v}_1), \dots T(\vec{v}_p)\}$ is linearly independent, then the original set $\{\vec{v}_1, \dots \vec{v}_p\}$ is linearly independent.

4.4 # 2, 3, 5, 7, 10, 11, 13, 15, 17, 21, 23, 32

2.) Find the vector $\vec{x}$ determined by the given coordinate vector $[\vec{x}]_{\mathcal{B}}$ and the given basis $\mathcal{B}$.

$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}, [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ $\quad \vec{x} = -2\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 5\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -26 \\ 1 \end{bmatrix}$

3.) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}, [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$

5.) Find the coordinate vector $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$ relative to the given basis $\mathcal{B} = \{\vec{b}_1, \dots \vec{b}_n\}$

$\vec{b}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad \left[\begin{array}{cc|c} 1 & 3 & -1 \\ -2 & -5 & 1 \end{array}\right] \xrightarrow{2R_1+R_2} \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{-3R_2+R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$ $\quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

7.) $\vec{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \vec{x} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{array}\right] \begin{matrix} \\ R_1+R_2 \\ 3R_1+R_3 \end{matrix} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 10 & 30 \end{array}\right] \begin{matrix} 3R_2+R_1 \\ \\ R_3/10 \end{matrix} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{-2R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$ $\quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$

10.) Find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}$ $\quad P_{\mathcal{B}} = [\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3] = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & -2 \\ 6 & -4 & 3 \end{bmatrix}$

11.) Use an inverse matrix to find $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$.

$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\vec{x} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \end{bmatrix}$

4.4

13.) The set $\mathcal{B} = \{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\vec{p}(t) = 1+4t+7t^2$ relative to $\mathcal{B}$.

$$c_1(1+t^2) + c_2(t+t^2) + c_3(1+2t+t^2) = 1+4t+7t^2$$

$$c_1 + c_3 + (c_2+2c_3)t + (c_1+c_2+c_3)t^2 = 1+4t+7t^2$$

$c_1 + c_3 = 1$
$c_2 + 2c_3 = 4$
$c_1+c_2+c_3 = 7$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 7 \end{array}\right] -R_1+R_3 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 6 \end{array}\right] -R_2+R_3 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -2 & 2 \end{array}\right] \begin{array}{l} R_3+R_2 \\ R_3/(-2) \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right] -R_3+R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad [\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$$

15.) True/False. $\mathcal{B}$ is a basis for a vector space V.

a.) If $\vec{x}$ is in V and if $\mathcal{B}$ contains n vectors, then the $\mathcal{B}$ coordinate vector of $\vec{x}$ is in $\mathbb{R}^n$.

b.) If $P_{\mathcal{B}}$ is the change-of-coordinates matrix, then $[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}\vec{x}$ for $\vec{x}$ in V.

c.) The vector spaces $\mathbb{P}_3$ and $\mathbb{R}^3$ are isomorphic.

a.) True b.) False c.) False

17.) The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ span $\mathbb{R}^2$ but do not form a basis. Find two different ways to express $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ -3 & -8 & 7 & 1 \end{array}\right] 3R_1+R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & -2 & -2 & 4 \end{array}\right] \begin{array}{l} R_2+R_1 \\ R_2/(-2) \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -5 & 5 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$x_1 = 5+5x_3$
$x_2 = -2-x_3$
x_3 free

infinitely many answers to the Problem

one answer: $\vec{x} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$ or $\vec{x} = \begin{bmatrix} 10 \\ -3 \\ 1 \end{bmatrix}$

4.4 continued

21.) Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \end{bmatrix} \right\}$. Since the coordinate mapping determined by $\mathcal{B}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$, this mapping must be implemented by some 2×2 matrix A. Find it. (Hint: Multiplication by A should transform a vector $\vec{x}$ into $[\vec{x}]_{\mathcal{B}}$.)

Since $P_{\mathcal{B}}^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}}$, $P_{\mathcal{B}}^{-1}$ is the matrix we are looking for.

$P_{\mathcal{B}} = \begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix}$ so $P_{\mathcal{B}}^{-1} = \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}$.

23.) V is a vector space, $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis and $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is the coordinate mapping. Show that the coordinate mapping is one-to-one. (Hint: Suppose $[\vec{u}]_{\mathcal{B}} = [\vec{w}]_{\mathcal{B}}$ for some $\vec{u}, \vec{w} \in V$ and show that $\vec{u} = \vec{w}$.)

$[\vec{u}]_{\mathcal{B}} = [\vec{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ So $\vec{u} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$ and $\vec{w} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$ therefore $\vec{u} = \vec{w}$. Thus the coordinate mapping is one-to-one.

32.) Let $\vec{p}_1(t) = 1 + t^2$, $\vec{p}_2(t) = t - 3t^2$, $\vec{p}_3(t) = 1 + t - 3t^2$

a.) Use coordinate vectors to show that these polynomials form a basis for $\mathbb{P}^2$.

b.) Consider the basis $\mathcal{B} = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ for $\mathbb{P}^2$. Find $\vec{q} \in \mathbb{P}^2$ s.t. $[\vec{q}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

a.) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ This is row equivalent to I_3, so its invertible, and so the columns are linearly independent and span $\mathbb{R}^3$. Since $\mathbb{R}^3$ is isomorphic to $\mathbb{P}^2$,

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \right\}$ forms a basis for $\mathbb{P}^2$.

b.) $\vec{q} = -1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -10 \end{bmatrix}$ $\quad \vec{q}(t) = 1 + 3t - 10t^2$