University of Connecticut
Department of Mathematics

Math 2210 — **Chapter 3 Worksheet** — Spring 2019

Due: Thursday, April 11

Directions: This worksheet covers Chapter 3 from Lay's Linear Algebra book. Turn in the red-squared problems below (problems 3, 5, 6, 10, 11, 12, 15, 20, 21, and 23). Use only one side of the paper, remove all fringes, and staple this cover sheet to your submission.

Any of these problems is fair game for the exam.
Ask questions if you're unsure about anything.

Group Member(s) (up to four)

Name: ______________________ Name: ______________________
Name: ______________________ Name: ______________________

Name **and** Section (of the student I will hand this back to):______________

Checklist for formatting:

- ☐ write every group member's name
- ☐ remove all fringes
- ☐ use only one side of the paper
- ☐ label each problem clearly
- ☐ give a clear justification for your answer
- ☐ circle your answer
- ☐ staple this coversheet to your submission

CHAPTER 3 SUPPLEMENTARY EXERCISES

1. Mark each statement True or False. Justify each answer. Assume that all matrices here are square.
   a. If A is a 2×2 matrix with a zero determinant, then one column of A is a multiple of the other.
   b. If two rows of a 3×3 matrix A are the same, then $\det A = 0$.
   c. If A is a 3×3 matrix, then $\det 5A = 5 \det A$.
   d. If A and B are $n \times n$ matrices, with $\det A = 2$ and $\det B = 3$, then $\det(A + B) = 5$.
   e. If A is $n \times n$ and $\det A = 2$, then $\det A^3 = 6$.
   f. If B is produced by interchanging two rows of A, then $\det B = \det A$.
   g. If B is produced by multiplying row 3 of A by 5, then $\det B = 5 \cdot \det A$.
   h. If B is formed by adding to one row of A a linear combination of the other rows, then $\det B = \det A$.
   i. $\det A^T = -\det A$.
   j. $\det(-A) = -\det A$.
   k. $\det A^TA \geq 0$.
   l. Any system of n linear equations in n variables can be solved by Cramer's rule.
   m. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^2$ and $\det[\,\mathbf{u}\ \ \mathbf{v}\,] = 10$, then the area of the triangle in the plane with vertices at $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ is 10.
   n. If $A^3 = 0$, then $\det A = 0$.
   o. If A is invertible, then $\det A^{-1} = \det A$.
   p. If A is invertible, then $(\det A)(\det A^{-1}) = 1$.

Use row operations to show that the determinants in Exercises 2–4 are all zero.

2. $\begin{vmatrix} 12 & 13 & 14 \\ 15 & 16 & 17 \\ 18 & 19 & 20 \end{vmatrix}$

3. $\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix}$

4. $\begin{vmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{vmatrix}$

Compute the determinants in Exercises 5 and 6.

5. $\begin{vmatrix} 9 & 1 & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 9 & 0 & 3 & 9 & 0 \\ 6 & 0 & 0 & 7 & 0 \end{vmatrix}$

6. $\begin{vmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{vmatrix}$

7. Show that the equation of the line in $\mathbb{R}^2$ through distinct points (x_1, y_1) and (x_2, y_2) can be written as
$$\det \begin{bmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix} = 0$$

8. Find a 3×3 determinant equation similar to that in Exercise 7 that describes the equation of the line through (x_1, y_1) with slope m.

Exercises 9 and 10 concern determinants of the following *Vandermonde matrices*.
$$T = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}, \quad V(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}$$

9. Use row operations to show that
$$\det T = (b-a)(c-a)(c-b)$$

10. Let $f(t) = \det V$, with x_1, x_2, and x_3 all distinct. Explain why $f(t)$ is a cubic polynomial, show that the coefficient of t^3 is nonzero, and find three points on the graph of f.

11. Find the area of the parallelogram determined by the points $(1, 4)$, $(-1, 5)$, $(3, 9)$, and $(5, 8)$. How can you tell that the quadrilateral determined by the points is actually a parallelogram?

12. Use the concept of area of a parallelogram to write a statement about a 2×2 matrix A that is true if and only if A is invertible.

13. Show that if A is invertible, then adj A is invertible, and
$$(\text{adj}\, A)^{-1} = \frac{1}{\det A} A$$
[*Hint:* Given matrices B and C, what calculation(s) would show that C is the inverse of B?]

14. Let A, B, C, D, and I be $n \times n$ matrices. Use the definition or properties of a determinant to justify the following formulas. Part (c) is useful in applications of eigenvalues (Chapter 5).
   a. $\det \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} = \det A$ b. $\det \begin{bmatrix} I & 0 \\ C & D \end{bmatrix} = \det D$
   c. $\det \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = (\det A)(\det D) = \det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$

15. Let A, B, C, and D be $n \times n$ matrices with A invertible.
   a. Find matrices X and Y to produce the block LU factorization
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & Y \end{bmatrix}$$
   and then show that
$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = (\det A) \cdot \det(D - CA^{-1}B)$$

b. Show that if $AC = CA$, then

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - CB)$$

16. Let J be the $n \times n$ matrix of all 1's, and consider $A = (a-b)I + bJ$; that is,

$$A = \begin{bmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{bmatrix}$$

Confirm that $\det A = (a-b)^{n-1}[a + (n-1)b]$ as follows:

a. Subtract row 2 from row 1, row 3 from row 2, and so on, and explain why this does not change the determinant of the matrix.

b. With the resulting matrix from part (a), add column 1 to column 2, then add this new column 2 to column 3, and so on, and explain why this does not change the determinant.

c. Find the determinant of the resulting matrix from (b).

17. Let A be the original matrix given in Exercise 16, and let

$$B = \begin{bmatrix} a-b & b & b & \cdots & b \\ 0 & a & b & \cdots & b \\ 0 & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b & b & \cdots & a \end{bmatrix},$$

$$C = \begin{bmatrix} b & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{bmatrix}$$

Notice that A, B, and C are nearly the same except that the first column of A equals the sum of the first columns of B and C. A *linearity property* of the determinant function, discussed in Section 3.2, says that $\det A = \det B + \det C$. Use this fact to prove the formula in Exercise 16 by induction on the size of matrix A.

18. [**M**] Apply the result of Exercise 16 to find the determinants of the following matrices, and confirm your answers using a matrix program.

$$\begin{bmatrix} 3 & 8 & 8 & 8 \\ 8 & 3 & 8 & 8 \\ 8 & 8 & 3 & 8 \\ 8 & 8 & 8 & 3 \end{bmatrix} \quad \begin{bmatrix} 8 & 3 & 3 & 3 & 3 \\ 3 & 8 & 3 & 3 & 3 \\ 3 & 3 & 8 & 3 & 3 \\ 3 & 3 & 3 & 8 & 3 \\ 3 & 3 & 3 & 3 & 8 \end{bmatrix}$$

19. [**M**] Use a matrix program to compute the determinants of the following matrices.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

Use the results to guess the determinant of the matrix below, and confirm your guess by using row operations to evaluate that determinant.

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$$

20. [**M**] Use the method of Exercise 19 to guess the determinant of

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 3 & \cdots & 3 \\ 1 & 3 & 6 & \cdots & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 3 & 6 & \cdots & 3(n-1) \end{bmatrix}$$

Justify your conjecture. [*Hint:* Use Exercise 14(c) and the result of Exercise 19.]