5.1 # 2, 6, 7, 11, 13, 15, 19, 21, 23, 24, 25, 27, 31

2.) Is $\lambda = -3$ an eigenvalue of $\begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$? Why or why not?

$\lambda = -3$ is an eigenvalue $\Leftrightarrow A\vec{x} = -3\vec{x}$ has a non trivial soln

$\Leftrightarrow (A+3I)\vec{x} = \vec{0}$ has a nontrivial soln

$$A+3I = \begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 6 & 12 & 0 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has free variable $\Rightarrow$ has nontrivial soln

So, $\boxed{\text{Yes}}$

6.) Is $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ an eigen vector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$? If so, find the eigenvalue.

Is $A\vec{x}$ a multiple of $\vec{x}$? $A\vec{x} = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix}$ This is not a multiple of $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ so $\boxed{\text{No}}$

7.) Is $\lambda = 4$ an eigen value of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector.

$$A-4I = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable $\Rightarrow$ has a nontrivial soln $\Rightarrow \lambda = 4$ is an eigenvalue.

Any nonzero solution to $(A-4I)\vec{x} = \vec{0}$ is an eigenvector.

$x_1 = -x_3$

$x_2 = -x_2$

x_3 free

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector. So is any multiple of $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

11.) Find a basis for the eigenspace corresponding to the eigenvalues

$A = \begin{bmatrix} 1 & -3 \\ -4 & 5 \end{bmatrix}, \lambda = -1, 7$

For $\lambda = -1$: $A + I = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$ $\left[\begin{array}{cc|c} 2 & -3 & 0 \\ -4 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = 3/2\, x_2$, x_2 free

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/2\, x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

A basis for the eigenspace corresponding to $\lambda = -1$ is $\left\{ \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \right\}$

11.) continued

For $\lambda = 7$: $A - 7I = \begin{bmatrix} -6 & -3 \\ -4 & -2 \end{bmatrix}$ $\left[\begin{array}{cc|c} -6 & -3 & 0 \\ -4 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = -\frac{1}{2}x_2$, x_2 free

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

A basis for the eigenspace corresponding to $\lambda = 7$ is $\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}$.

13.) Same directions as in #11.

$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $\lambda = 1, 2, 3$

For $\lambda = 1$: $A - I = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

So the solutions to $(A - I)\vec{x} = \vec{0}$ are $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

A basis for the eigenspace is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 2$: $A - 2I = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

So the solutions to $(A - 2I)\vec{x} = \vec{0}$ are $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$

A basis for the eigenspace is $\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = 3$: $A - 3I = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

So the solns to $(A - 3I)\vec{x} = \vec{0}$ are $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

A basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

5.1 Continued

15.) $A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}$, $\lambda = -5$ $\quad A + 5I = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 + x_2 + x_3 = 0$, x_2, x_3 free

So solutions to $(A+5I)\vec{x} = \vec{0}$ are $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

A basis for the eigenspace corresponding to $\lambda = -5$ is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

19.) For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, find one eigenvalue, with no calculation. Justify your answer.

The columns of A are linearly dependent, so A is not invertible by IMT. Therefore zero is an eigenvalue of A.

21.) True/False. Let A be an $n \times n$ matrix.

a.) If $A\vec{x} = \lambda \vec{x}$ for some vector $\vec{x}$, then λ is an eigen value of A.

b.) A matrix A is not invertible iff zero is an eigenvalue of A.

c.) A number c is an eigenvalue of A iff the equation $(A - cI)\vec{x} = \vec{0}$ has a nontrivial soln.

d.) Finding an eigen vector of A may be difficult, but checking whether a given vector is an eigenvector is easy.

e.) To find the eigenvalues of A, reduce A to echelon form.

a.) False b.) True c.) True d.) True e.) False

23.) Explain why a 2×2 matrix can have at most two distinct eigen values. Explain why an $n \times n$ matrix can have at most n distinct eigen values.

The set of eigenvectors corresponding to distinct eigenvalues is linearly independent by Thm 2. For an $n \times n$ matrix, the eigen vectors have n entries. Having more than n distinct eigen values means you have more than n eigen vectors. A set with more vectors than entries in each vector is linearly dependent.

24.) Construct an example of a 2x2 matrix with only one distinct eigenvalue. The eigenvalues of a triangular matrix are the entries on the main diagonal. So choose any 2x2 triangular matrix with the same number on its diagonal.

$$\underset{\lambda=3}{\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}}, \quad \underset{\lambda=-4}{\begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}}, \quad \text{etc...}$$

25.) Let λ be an eigenvalue of an invertible matrix A. Show λ^{-1} is an eigenvalue of A^{-1}. If λ is an eigenvalue of A then by defn there is a nonzero vector $\vec{x}$ s.t. $A\vec{x} = \lambda\vec{x}$. Since A is invertible, we can apply A^{-1} to both sides of the equation, to get $\underbrace{A^{-1}A}_{\text{identity}}\vec{x} = A^{-1}\lambda\vec{x} \Rightarrow \vec{x} = A^{-1}\lambda\vec{x}$

(where $\vec{x}$ is this nonzero vector)

Since λ is a scalar, we can rearrange, $\vec{x} = \lambda A^{-1}\vec{x}$ and then multiply both sides by λ^{-1}. $\lambda^{-1}\vec{x} = \underbrace{\lambda^{-1}\lambda}_{1} A^{-1}\vec{x} \Rightarrow \lambda^{-1}\vec{x} = A^{-1}\vec{x}$

Since $\vec{x}$ is non zero, λ^{-1} is an eigenvalue of A^{-1}.

27.) Show that λ is an eigen value of A if and only if λ is an eigenvalue of A^T.

λ is an eigen value of $A \Leftrightarrow (A-\lambda I)\vec{x} = \vec{0}$ has a nontrivial soln.

$(A-\lambda I)\vec{x} = \vec{0}$ has a non trivial soln $\Leftrightarrow A-\lambda I$ is not invertible (IMT)

$A-\lambda I$ is not invertible $\Leftrightarrow (A-\lambda I)^T$ is not invertible. $\Leftrightarrow A^T-\lambda I$ is not invertible

$(A-\lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I$ $\quad\Leftrightarrow \lambda$ is an eignenvalue of A^T

31.) A is the standard matrix of T. T is the transformation on $\mathbb{R}^2$ that reflects points across some line through the origin. Without writing A, find an eigenvalue of A and describe the eigen space. A line through the origin is the set of all multiples of some nonzero vector $\vec{v}$. T reflects points across this line, so points <u>on</u> the line stay put, meaning $T(\vec{v}) = \vec{v}$ or in terms of the matrix $A\vec{v} = \vec{v}$. $\vec{v}$ is nonzero, so its an eigenvector corresponding to $\lambda = 1$.

The eigenspace corresponding to $\lambda = 1$ is Span$\{\vec{v}\}$.

5.2 # 2, 5, 9, 12, 15, 19, 20, 21

2.) Find the characteristic polynomial and the real eigenvalues of the matrix.

$\begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$ Characteristic polynomial $= \det(A-\lambda I) = \begin{vmatrix} -4-\lambda & -1 \\ 6 & 1-\lambda \end{vmatrix} =$

$= (-4-\lambda)(1-\lambda)+6 = \boxed{\lambda^2+3\lambda+2} = (\lambda+1)(\lambda+2)$

λ is an eigenvalue of A iff $\det(A-\lambda I)=0$

$(\lambda+1)(\lambda+2)=0 \Rightarrow \boxed{\lambda=-1,\ \lambda=-2}$

5.) $\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$ $\det(A-\lambda I) = (8-\lambda)(8-\lambda)-16 = \boxed{\lambda^2-16\lambda+48}$

set equal to zero: $\lambda^2-16\lambda+48 = (\lambda-4)(\lambda-12)=0$ $\boxed{\lambda = 4, 12}$

9.) Find the characteristic polynomial of $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix}$.

(cofactor expansion 2^{nd} column)

$\begin{vmatrix} 4-\lambda & 0 & -1 \\ 0 & 4-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (4-\lambda)\left[(4-\lambda)(2-\lambda)+1\right] = (4-\lambda)\left[\lambda^2-6\lambda+9\right]$

$= 4\lambda^2-24\lambda+36-\lambda^3+6\lambda^2-9\lambda = \boxed{-\lambda^3+10\lambda^2-33\lambda+36}$

12.) $\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ $\begin{vmatrix} -1-\lambda & 0 & 2 \\ 3 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{vmatrix} \overset{(1^{st}\text{ row})}{=} (-1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} + 2\begin{vmatrix} 3 & 1-\lambda \\ 0 & 1 \end{vmatrix}$

$= (-1-\lambda)(1-\lambda)(2-\lambda)+6 = \boxed{-\lambda^3+2\lambda^2+\lambda+4}$

15.) List the real eigenvalues repeated according to their multiplicities.

$\begin{bmatrix} 5 & 5 & 0 & 2 \\ 0 & 2 & -3 & 6 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ A$-\lambda$I is also triangular, so $\det(A-\lambda I)$ is the product of its diagonal entries.

triangular

$\det(A-\lambda I) = (5-\lambda)(2-\lambda)(3-\lambda)(5-\lambda)$

$\boxed{\lambda = 5, 5, 2, 3}$

19.) Let A be an $n \times n$ matrix, and suppose A has n real eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, repeated according to multiplicities, so that $\det(A-\lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\ldots(\lambda_n - \lambda)$. Explain why $\det(A)$ is the product of the n eigenvalues of A.

This is obvious if A is triangular, but we are not assuming that here.

$\det(A) = \det(A-\lambda I)$ when $\lambda = 0$. Therefore $\det(A) = (\lambda_1 - 0)(\lambda_2 - 0)\cdots(\lambda_n - 0)$
$= \lambda_1 \lambda_2 \ldots \lambda_n$.

20.) Use a property of determinants to show that A and A^T have the same characteristic polynomial.

We know $\det(A) = \det(A^T)$ for all square matrices. So $\det(A-\lambda I) = \det((A-\lambda I)^T)$
$= \det(A^T - \lambda I^T) = \det(A^T - \lambda I)$.

21.) True/False. A and B are $n \times n$ matrices.

a.) The determinant of A is the product of the diagonal entries in A.

b.) An elementary row operation on A doesn't change the determinant.

c.) $(\det A)(\det B) = \det(AB)$

d.) If $\lambda + 5$ is a factor of the characteristic polynomial of A, then 5 is an eigenvalue of A.

a.) False
b.) False
c.) True

These are all review from previous sections

d.) False (-5 is an eigenvalue)

5.3 # 1, 4, 5, 9, 11, 15, 17, 21, 24, 26

1.) Let $A = PDP^{-1}$ and compute A^4. $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$A^4 = (PDP^{-1})^4 = PD\underbrace{P^{-1}P}_{I}D\underbrace{P^{-1}P}_{I}D\underbrace{P^{-1}P}_{I}DP^{-1} = PD^4P^{-1}$, $D^4 = \begin{bmatrix} 2^4 & 0 \\ 0 & 1^4 \end{bmatrix}$ since D is diagonal

$$A^4 = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}$$

4.) $A = PDP^{-1}$ is given. Compute A^k where k is an arbitrary positive integer.

$$\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \qquad A^k = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} (-3)^k & 0 \\ 0 & (-2)^k \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -3(-3)^k + 4(-2)^k & 6(-3)^k - 6(-2)^k \\ -2(-3)^k + 2(-2)^k & 4(-3)^k - 3(-2)^k \end{bmatrix}$$

5.) $A = PDP^{-1}$ is given. Use the diagonalization thm to find the eigenvalues of A and a basis for each eigenspace.

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

The diagonal entries of D are eigenvalues corresponding to the eigenvectors given by the columns of p respectively. So $\lambda = 2$ corresponds to $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ and $\lambda = 3$ corresponds to both $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. So $\left\{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}\right\}$ is a basis for the eigenspace for $\lambda = 2$ and $\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace for $\lambda = 3$.

9.) Diagonalize the matrix if possible.

$\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$ $\det(A - \lambda I) = (2-\lambda)(4-\lambda) + 1 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$ So $\lambda = 3$

$A - 3I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ $(A - 3I)\vec{x} = \vec{0}$ $\left[\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{cc|c} +1 & +1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = -x_2$, x_2 free

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ So $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace, but A is not diagonalizable because A has only 1 linearly independent eigenvector. It needs exactly 2 because A is 2x2.

11.) $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{bmatrix}$ $\lambda = -1, 5$

For $\lambda = -1$
$(A+I)\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = -x_2 - x_3$, x_2, x_3 free

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ So $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace.

For $\lambda = 5$
$(A-5I)\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} -5 & 1 & 1 & 0 \\ 2 & -4 & 2 & 0 \\ 3 & 3 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$ So $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a basis for the eigenspace

I took a multiple of $\begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$ to get rid of the fractions

Since $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ are linearly independent, (check this if not obvious)

we can let $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

15.) $\begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ $\lambda = 0, 1$

For $\lambda = 0$
$(A+0I)\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = x_3$, $x_2 = -x_3$, x_3 free $\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

For $\lambda = 1$
$(A-I)\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = -x_2 - x_3$, x_2, x_3 free $\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

17.) $\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$ This matrix is triangular, $\lambda = 2$.

$(A-2I)\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = 0$, $x_2 = 0$, x_3 free $\vec{x} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the 1-dimensional eigenspace. Since we can find at most 1 lin. indep eigenvector (and not exactly 3), A is not diagonizable.

5.3 continued

21.) True/False. A, B, P, D are all $n \times n$ matrices.

a.) A is Diagonalizable if $A = PDP^{-1}$ for some matrix D and some invertible matrix P.

b.) If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.

c.) A is diagonalizable iff A has n eigenvalues, counting multiplicities.

d.) If A is diagonalizable, then A is invertible.

a.) False (only if D is diagonal) b.) True c.) False d.) False.

24.) A is a 3×3 matrix with two eigen values. Each eigenspace is one-dimensional. Is A diagonalizable? why?

No, if each eigenspace is one-dimensional, we can have at most 2 Linearly independent eigenvectors. A 3×3 matrix needs exactly 3 linearly independent eigenvectors to be diagonalizable.

26.) A is a 7×7 matrix with three eigenvalues. One eigenspace is two dimensional, and one of the other eigen spaces is three (3) dimensional. is it possible that A is not diagonalizable? Justify.

Yes, if the third eigenspace is only 1-dimensional then we could have at most $2+3+1 = 6$ linearly indep. eigenvectors and A would not be diagonalizable.

5.4 # 1, 3, 6, 7, 10, 15, 16, 23, 25

1.) Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $\mathcal{D} = \{\vec{d}_1, \vec{d}_2\}$ be bases for V and W respectively. Let $T: V \to W$ be a linear transformation s.t $T(\vec{b}_1) = 3\vec{d}_1 - 5\vec{d}_2$, $T(\vec{b}_2) = -\vec{d}_1 + 6\vec{d}_2$ and $T(\vec{b}_3) = 4\vec{d}_2$. Find the matrix for T relative to $\mathcal{B}$ and $\mathcal{D}$.

$$\begin{bmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \end{bmatrix}$$

3.) Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis for $\mathbb{R}^3$, let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ be a basis for a vector space V, and let $T: \mathbb{R}^3 \to V$ be a linear transformation s.t $T(x_1, x_2, x_3) = (2x_3 - x_2)\vec{b}_1 - (2x_2)\vec{b}_2 + (x_1 + 3x_3)\vec{b}_3$

a.) Compute $T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)$

$T(\vec{e}_1) = T(1,0,0) = 0\vec{b}_1 + 0\vec{b}_2 + \vec{b}_3$

$T(\vec{e}_2) = T(0,1,0) = \vec{b}_1 - 2\vec{b}_2 + 0\vec{b}_3$

$T(\vec{e}_3) = T(0,0,1) = 2\vec{b}_1 + 0\vec{b}_2 + 3\vec{b}_3$

b.) Compute $[T(\vec{e}_1)]_{\mathcal{B}}, [T(\vec{e}_2)]_{\mathcal{B}}, [T(\vec{e}_3)]_{\mathcal{B}}$

$[T(\vec{e}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $[T(\vec{e}_2)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $[T(\vec{e}_3)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

c.) Compute the matrix for T relative to $\mathcal{E}$ an $\mathcal{B}$.

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

6.) Let $T: \mathbb{P}_2 \to \mathbb{P}_4$ be the transformation that maps a polynomial $\vec{p}(t)$ into the polynomial $\vec{p}(t) + 2t^2\vec{p}(t)$.

a.) Find the image of $\vec{p}(t) = 3 - 2t + t^2$. $3 - 2t + t^2 + 2t^2(3 - 2t + t^2)$

$= 2t^4 - 4t^3 + 7t^2 - 2t + 3$

b.) Show that T is a linear transformation.

Let $\vec{p}$ & $\vec{q}$ be polynomials in $\mathbb{P}_2$ and c be scalar.

$cT(\vec{p}) = c(\vec{p} + 2t^2\vec{p}) = c\vec{p} + 2ct^2\vec{p}$ and $T(c\vec{p}) = c\vec{p} + 2t^2(c\vec{p}) = c\vec{p} + 2ct^2\vec{p}$ ✓

$T(\vec{p}+\vec{q}) = (\vec{p}+\vec{q}) + 2t^2(\vec{p}+\vec{q})$ and $T(\vec{p}) + T(\vec{q}) = \vec{p} + 2t^2\vec{p} + \vec{q} + 2t^2\vec{q}$ ✓

$= \vec{p} + \vec{q} + 2t^2\vec{p} + 2t^2\vec{q}$

c.) Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$. $\mathcal{C}$

$T(1) = 1 + 2t^2$ $T(t) = t + 2t^3$ $T(t^2) = t^2 + 2t^4$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$[T(1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

7.) Assume the mapping $T: \mathbb{P}_2 \to \mathbb{P}_2$ defined by $T(a_0 + a_1 t + a_2 t^2) = 3a_0 + (5a_0 - 2a_1)t + (4a_1 + a_2)t^2$ is linear. Find the matrix representation of T relative to the basis $\mathcal{B} = \{1, t, t^2\}$.

$T(1) = 3 + 5t \quad T(t) = -2t + 4t^2 \quad T(t^2) = t^2$

$[T(1)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} \quad [T(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} \quad T(t^2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

10.) Define $T: \mathbb{P}_3 \to \mathbb{R}^4$ by $T(\vec{p}) = \begin{bmatrix} \vec{p}(-2) \\ \vec{p}(3) \\ \vec{p}(1) \\ \vec{p}(0) \end{bmatrix}$.

a.) Show that T is a lin. trans.

b.) Find the matrix of T relative to the basis $\{1, t, t^2, t^3\}$ for $\mathbb{P}_3$ and the standard basis for $\mathbb{R}^4$.

a.) $cT(\vec{p}) = \begin{bmatrix} c\vec{p}(-2) \\ c\vec{p}(3) \\ c\vec{p}(1) \\ c\vec{p}(0) \end{bmatrix} = T(c\vec{p})$ ✓

$T(\vec{p} + \vec{q}) = \begin{bmatrix} (\vec{p}+\vec{q})(-2) \\ (\vec{p}+\vec{q})(3) \\ (\vec{p}+\vec{q})(1) \\ (\vec{p}+\vec{q})(0) \end{bmatrix} = \begin{bmatrix} \vec{p}(-2) + \vec{q}(-2) \\ \vec{p}(3) + \vec{q}(3) \\ \vec{p}(1) + \vec{q}(1) \\ \vec{p}(0) + \vec{q}(0) \end{bmatrix}$ ✓

$T(\vec{p}) + T(\vec{q}) = \begin{bmatrix} \vec{p}(-2) \\ \vec{p}(3) \\ \vec{p}(1) \\ \vec{p}(0) \end{bmatrix} + \begin{bmatrix} \vec{q}(-2) \\ \vec{q}(3) \\ \vec{q}(1) \\ \vec{q}(0) \end{bmatrix} = \begin{bmatrix} \vec{p}(-2) + \vec{q}(-2) \\ \vec{p}(3) + \vec{q}(3) \\ \vec{p}(1) + \vec{q}(1) \\ \vec{p}(0) + \vec{q}(0) \end{bmatrix}$

b.) $T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, T(t) = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, T(t^2) = \begin{bmatrix} 4 \\ 9 \\ 1 \\ 0 \end{bmatrix}, T(t^3) = \begin{bmatrix} -8 \\ 27 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & 3 & 9 & 27 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

5.4 Continued

15.) Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = A\vec{x}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the Property that $[T]_{\mathcal{B}}$ is diagonal.

$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$. Diagonalize A. The characteristic polynomial is $(1-\lambda)(-4-\lambda)-6 =$
$= \lambda^2 + 3\lambda - 10 = (\lambda+5)(\lambda-2)$ so $-5, 2$ are eigenvalues

$\lambda = -5$
$(A+5I)\vec{x} = \vec{0}$ $\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = -\frac{1}{3}x_2$, x_2 free $\vec{x} = x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$ Use $\left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$ as a basis for the eigenspace.

$\lambda = 2$
$(A-2I)\vec{x} = \vec{0}$ $\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = 2x_2$, x_2 free $\vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$A = PDP^{-1}$ where $P = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$ $\mathcal{B} = \left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ Thm 8

16.) $A = \begin{bmatrix} 4 & -2 \\ -1 & 5 \end{bmatrix}$ $(4-\lambda)(5-\lambda) - 2 = \lambda^2 - 9\lambda + 18 = (\lambda-3)(\lambda-6)$, $\lambda = 3, 6$

$\lambda = 3$
$(A-3I)\vec{x} = \vec{0}$ $\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = 2x_2$, x_2 free $\vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\mathcal{B} = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$\lambda = 6$
$(A-6I)\vec{x} = \vec{0}$ $\left[\begin{array}{cc|c} -2 & -2 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = -x_2$, x_2 free $\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

23.) If $B = P^{-1}AP$ and $\vec{x}$ is an eigenvector of A corresponding to an eigenvalue λ, then $P^{-1}\vec{x}$ is an eigenvector of B corresponding to λ.
Verify. We want to show $B(P^{-1}\vec{x}) = \lambda(P^{-1}\vec{x})$

$$BP^{-1}\vec{x} = (P^{-1}A\underbrace{P)P^{-1}}_{I}\vec{x} = P^{-1}A\vec{x}$$ since $\vec{x}$ is an eigenvector of A, $A\vec{x} = \lambda\vec{x}$
$$= P^{-1}\lambda\vec{x}$$
$$= \lambda P^{-1}\vec{x}$$

25.) The trace of a square matrix A is the sum is the sum of the diagonal entries of A and is denoted by tr A.
It can be verified that $tr(FG) = tr(GF)$ for any two $n \times n$ matrices F and G. Show that if A, and B are similar then $tr A = tr B$.

A is similar to B if there exists an invertible matrix P s.t. $A = PBP^{-1}$.

$$tr A = tr(PBP^{-1}) = tr(P^{-1}(PB)) = tr(IB) = tr B.$$

6.1 # 3, 5, 10, 16, 18, 19, 23, 25, 27, 29

3.) Let $\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$ Compute:

$\dfrac{1}{\vec{w} \cdot \vec{w}} \vec{w}$

$$\vec{w} \cdot \vec{w} = \vec{w}^T \vec{w} = \begin{bmatrix} 3 & -1 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 3(3) - 1(-1) - 5(-5) = 9 + 1 + 25 = 35$$

$$\frac{1}{\vec{w} \cdot \vec{w}} \vec{w} = \frac{1}{35} \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 3/35 \\ -1/35 \\ -1/7 \end{bmatrix}$$

5.) $\left(\dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right) \vec{v}$

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = -4 + 12 = 8, \quad \vec{v} \cdot \vec{v} = \begin{bmatrix} 4 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 16 + 36 = 52$$

$$\left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right) \vec{v} = \left(\frac{8}{52}\right) \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \left(\frac{2}{13}\right) \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8/13 \\ 12/13 \end{bmatrix}$$

10.) Find a unit vector in the direction of $\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$.

$$\vec{u} = \frac{1}{\sqrt{(-6)^2 + 4^2 + (-3)^2}} \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{61}} \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}$$

16.) Determine if the vectors are orthogonal.

$\vec{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} 12 & 3 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = 24 - 9 - 15 = 0$$

Since $\vec{u} \cdot \vec{v} = 0$, $\vec{u}$ and $\vec{v}$ are orthogonal.

18.) $\vec{y} = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

$$\vec{y} \cdot \vec{z} = \begin{bmatrix} -3 & 7 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix} = -3 - 56 + 60 + 0 = 1$$

Since $\vec{y} \cdot \vec{z} \neq 0$, $\vec{y}$ and $\vec{z}$ are <u>not</u> orthogonal.

19.) True/False. All vectors are in $\mathbb{R}^n$.

a.) $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

b.) For any scalar c, $\vec{u} \cdot (c\vec{v}) = c(\vec{u} \cdot \vec{v})$.

c.) If the distance from $\vec{u}$ to $\vec{v}$ equals the distance from $\vec{u}$ to $-\vec{v}$, then $\vec{u}$ and $\vec{v}$ are orthogonal

d.) For a square matrix A, vectors in Col A are orthogonal to vectors in Nul A.

e.) If vectors $\vec{v}_1, \ldots \vec{v}_p$ span a subspace W and if $\vec{x}$ is orthogonal to each $\vec{v}_j$ for $j = 1, \ldots, p$, then $\vec{x}$ is in $W^{\perp}$.

a.) True b.) True c.) True d.) False e.) True

23.) Let $\vec{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}$. Compute and compare $\vec{u} \cdot \vec{v}$, $\|\vec{u}\|^2$, $\|\vec{v}\|^2$ and $\|\vec{u}+\vec{v}\|^2$. Do not use the pythagorean thm.

$\vec{u} \cdot \vec{v} = -14 + 20 - 6 = 0$

$\|\vec{v}\|^2 = \vec{v} \cdot \vec{v} = (-7)^2 + (-4)^2 + 6^2 = 101$

$\|\vec{u}\|^2 = \vec{u} \cdot \vec{u} = 2^2 + (-5)^2 + (-1)^2 = 30$

$\|\vec{u}+\vec{v}\|^2 = (-5)^2 + (-9)^2 + (5)^2 = 131$

25.) Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. Describe the set H of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\vec{v}$.

Hint: Consider $\vec{v} = \vec{0}$ and $\vec{v} \neq 0$.

$\begin{bmatrix} x \\ y \end{bmatrix}$ is orthogonal to $\vec{v}$ if the dot product equals zero, that is, if $ax + by = 0$. If $\vec{v} = \vec{0}$ then $0x + 0y = 0$ for any values of x, y. In this case, $H = \mathbb{R}^2$. If $\vec{v} \neq 0$, then either $a \neq 0$ or $b \neq 0$. Suppose $a \neq 0$. Then $ax + by = 0 \iff ax = -by \iff x = \left(\frac{-b}{a}\right)y$.

Then $H = \left\{ \begin{bmatrix} \frac{-b}{a}y \\ y \end{bmatrix} : y \in \mathbb{R} \right\}$. H has basis $\left\{ \begin{bmatrix} -b/a \\ 1 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} -b \\ a \end{bmatrix} \right\}$.

6.1 continued

27.) Suppose a vector $\vec{y}$ is orthogonal to vectors $\vec{u}$ and $\vec{v}$. Show that $\vec{y}$ is orthogonal to $\vec{u}+\vec{v}$.

We know that $\vec{y}\cdot\vec{u}=0$ and $\vec{y}\cdot\vec{v}=0$. We want to show that $\vec{y}\cdot(\vec{u}+\vec{v})=0$.

$$\begin{aligned}\vec{y}\cdot(\vec{u}+\vec{v}) &= \vec{y}\cdot\vec{u}+\vec{y}\cdot\vec{v}\\ &= 0+0\\ &= 0\end{aligned}$$

Thus $\vec{y}$ is orthogonal to $\vec{u}+\vec{v}$

29.) Let $W=\text{Span}\{\vec{v}_1,\dots,\vec{v}_p\}$. Show that if $\vec{x}$ is orthogonal to each $\vec{v}_j$, for $1\le j\le p$, then $\vec{x}$ is orthogonal to every vector in W.

Suppose $\vec{x}$ is orthogonal to each $\vec{v}_j$. Then $\vec{x}\cdot\vec{v}_j=0$ for each $\vec{v}_j$.

Any vector $\vec{w}$ in W is of the form $\vec{w}=c_1\vec{v}_1+\dots+c_p\vec{v}_p$

then
$$\begin{aligned}\vec{x}\cdot\vec{w} &= \vec{x}\cdot(c_1\vec{v}_1+\dots+c_p\vec{v}_p)\\ &= \vec{x}\cdot(c_1\vec{v}_1)+\dots+\vec{x}\cdot(c_p\vec{v}_p)\\ &= c_1(\vec{x}\cdot\vec{v}_1)+\dots+c_p(\vec{x}\cdot\vec{v}_p)\\ &= c_1(0)+\dots+c_p(0)\\ &= 0\end{aligned}$$

Since $\vec{x}\cdot\vec{w}=0$, $\vec{x}$ is orthogonal to $\vec{w}$ for any $\vec{w}$ in W.

6.2 # 3, 6, 8, 9, 11, 14, 20, 21, 23, 26, 27, 28, 29

3.) Determine if the set of vectors is orthogonal.

$$\left\{\begin{bmatrix}2\\-7\\-1\end{bmatrix}, \begin{bmatrix}-6\\-3\\9\end{bmatrix}, \begin{bmatrix}3\\1\\-1\end{bmatrix}\right\} \quad \begin{bmatrix}2\\-7\\-1\end{bmatrix}\cdot\begin{bmatrix}-6\\-3\\9\end{bmatrix} = -12+21-9=0 \quad \begin{bmatrix}2\\-7\\-1\end{bmatrix}\cdot\begin{bmatrix}3\\1\\-1\end{bmatrix} = 6-7+1=0$$

$$\begin{bmatrix}-6\\-3\\9\end{bmatrix}\cdot\begin{bmatrix}3\\1\\-1\end{bmatrix} = -18-3-9=-30 \neq 0$$

The set is not orthogonal.

6.) $$\left\{\begin{bmatrix}5\\-4\\0\\3\end{bmatrix}, \begin{bmatrix}-4\\1\\-3\\8\end{bmatrix}, \begin{bmatrix}3\\3\\5\\-1\end{bmatrix}\right\} \quad \begin{bmatrix}5\\-4\\0\\3\end{bmatrix}\cdot\begin{bmatrix}-4\\1\\-3\\8\end{bmatrix} = -20-4+0+24=0 \quad \begin{bmatrix}5\\-4\\0\\3\end{bmatrix}\cdot\begin{bmatrix}3\\3\\5\\-1\end{bmatrix} = 15-12+0-3=0$$

$$\begin{bmatrix}-4\\1\\-3\\8\end{bmatrix}\cdot\begin{bmatrix}3\\3\\5\\-1\end{bmatrix} = -12+3-15-8=-32 \neq 0$$

The set is not orthogonal

8.) Show that $\{u_1, u_2\}$ or $\{u_1, u_2, u_3\}$ is an orthogonal basis for $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively. Then express $\vec{x}$ as a linear combination of the $\vec{u}$'s.

$\vec{u}_1 = \begin{bmatrix}3\\1\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}-2\\6\end{bmatrix}, \vec{x} = \begin{bmatrix}-6\\3\end{bmatrix}$ $\quad \vec{u}_1\cdot\vec{u}_2 = -6+6=0$, So $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set.

Since $\vec{u}_1, \vec{u}_2$ are non-zero, the set is linearly independent by Thm 4. Since $\dim \mathbb{R}^2 = 2$ and our set is lin. indep consisting of exactly 2 vectors, it is a basis of $\mathbb{R}^2$. Thus we have an orthogonal basis.

$\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2$ where $c_1 = \dfrac{\vec{x}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = -\dfrac{3}{2}$ and $c_2 = \dfrac{\vec{x}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \dfrac{3}{4}$

$$\vec{x} = \frac{-3}{2}\vec{u}_1 + \frac{3}{4}\vec{u}_2$$

9.) Same directions as #8.

$u_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $u_2 = \begin{bmatrix}-1\\4\\1\end{bmatrix}$, $u_3 = \begin{bmatrix}2\\1\\-2\end{bmatrix}$, $\vec{x} = \begin{bmatrix}8\\-4\\3\end{bmatrix}$

$\vec{u}_1 \bullet \vec{u}_2 = -1+1=0$

$\vec{u}_1 \bullet \vec{u}_3 = 2-2=0$

$\vec{u}_2 \bullet \vec{u}_3 = -2+4-2=0$

$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set.

Since the set is orthogonal and doesn't contain $\vec{0}$, it is lin. indep.
Since $\dim \mathbb{R}^3 = 3$, the set is a basis for $\mathbb{R}^3$.

$\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$ where $c_1 = \dfrac{\vec{x}\bullet\vec{u}_1}{\vec{u}_1\bullet\vec{u}_1} = \dfrac{8+3}{1+1} = \dfrac{11}{2}$

$$c_2 = \frac{\vec{x}\bullet\vec{u}_2}{\vec{u}_2\bullet\vec{u}_2} = \frac{-8-16+3}{1+16+1} = \frac{-21}{18} = \frac{-7}{6}$$

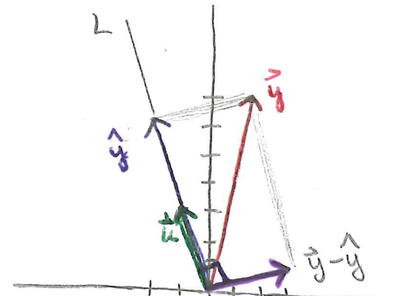
$$c_3 = \frac{\vec{x}\bullet\vec{u}_3}{\vec{u}_3\bullet\vec{u}_3} = \frac{16-4-6}{4+1+4} = \frac{6}{9} = 2/3$$

$$\boxed{\vec{x} = \frac{11}{2}\vec{u}_1 - \frac{7}{6}\vec{u}_2 + \frac{2}{3}\vec{u}_3}$$

11.) Compute the orthogonal projection of $\vec{y} = \begin{bmatrix}1\\7\end{bmatrix}$ onto the line through $\vec{u} = \begin{bmatrix}-1\\3\end{bmatrix}$ and the origin.

L is the line through $\vec{u} = \begin{bmatrix}-1\\3\end{bmatrix}$ and the origin.

$$\hat{y} = \text{Proj}_L \vec{y} = \left(\frac{\vec{y}\bullet\vec{u}}{\vec{u}\bullet\vec{u}}\right)\vec{u} = \left(\frac{-1+21}{1+9}\right)\begin{bmatrix}-1\\3\end{bmatrix} = 2\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}-2\\6\end{bmatrix}$$



Notice that $\vec{y}-\hat{y}$ is orthogonal to $\vec{u}$

* I copied the problem wrong, so this isn't exactly # 11, but the solution is correct for the problem I wrote.

14.) Let $\vec{y} = \begin{bmatrix}2\\6\end{bmatrix}$ and $\vec{u} = \begin{bmatrix}7\\1\end{bmatrix}$. Write $\vec{y}$ as the sum of a vector in Span$\{\vec{u}\}$ and a vector orthogonal to $\vec{u}$.

$$\hat{y} = \text{Proj}_L \vec{y} = \left(\frac{\vec{y}\bullet\vec{u}}{\vec{u}\bullet\vec{u}}\right)\vec{u} = \left(\frac{14+6}{49+1}\right)\begin{bmatrix}7\\1\end{bmatrix} = \begin{bmatrix}14/5\\2/5\end{bmatrix}$$

$\hat{y}$ is in Span$\{\vec{u}\}$ ($\hat{y}$ is a scalar multiple of $\vec{u}$)

$\vec{y} - \hat{y} = \begin{bmatrix}2\\6\end{bmatrix} - \begin{bmatrix}14/5\\2/5\end{bmatrix} = \begin{bmatrix}-4/5\\28/5\end{bmatrix}$ This is orthogonal to $\vec{u}$ because

$$\begin{bmatrix}-4/5\\28/5\end{bmatrix}\bullet\begin{bmatrix}7\\1\end{bmatrix} = \frac{-28}{5} + \frac{28}{5} = 0$$

$$\boxed{\vec{y} = \begin{bmatrix}14/5\\2/5\end{bmatrix} + \begin{bmatrix}-4/5\\28/5\end{bmatrix}}$$

6.2 Continued

20.) Determine if the set is orthonormal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

$$\left\{ \underset{\vec{v}}{\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}}, \underset{\vec{u}}{\begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}} \right\}$$

$\vec{v} \cdot \vec{u} = -\frac{2}{9} + \frac{2}{9} + 0 = 0$ So, these vectors are orthogonal.

$\|\vec{v}\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1$ So, $\vec{v}$ is a unit vector

$\|\vec{u}\| = \sqrt{\frac{1}{9} + \frac{4}{9} + 0} = \sqrt{5}/3$ $\vec{u}$ is not a unit vector

The set given is not orthonormal, but we can normalize $\vec{u}$.

$\frac{1}{\|\vec{u}\|}\vec{u} = \frac{3}{\sqrt{5}}\begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$. The set $\left\{ \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \right\}$ is orthonormal.

21.) $$\left\{ \underset{\vec{v}}{\begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}}, \underset{\vec{u}}{\begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}}, \underset{\vec{x}}{\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}} \right\}$$

$\vec{v} \cdot \vec{u} = \frac{3}{10} + \frac{-3}{20} + \frac{-3}{20} = 0$

$\vec{v} \cdot \vec{x} = 0 + \frac{-3}{\sqrt{2}\sqrt{20}} + \frac{3}{\sqrt{2}\sqrt{20}} = 0$

$\vec{u} \cdot \vec{x} = 0 + \frac{1}{\sqrt{2}\sqrt{20}} + \frac{-1}{\sqrt{2}\sqrt{20}} = 0$

The set is orthogonal.

$\|\vec{v}\| = \sqrt{\frac{1}{10} + \frac{9}{20} + \frac{9}{20}} = 1$ $\quad \|\vec{u}\| = \sqrt{\frac{9}{10} + \frac{1}{20} + \frac{1}{20}} = 1$ $\quad \|\vec{x}\| = \sqrt{0 + \frac{1}{2} + \frac{1}{2}} = 1$

The set is orthonormal.

23.) True/False. All vectors are in $\mathbb{R}^n$.

a) Not every linearly independent set in $\mathbb{R}^n$ is an orthogonal set.

b.) If $\vec{y}$ is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix.

c.) If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.

d.) A matrix with orthonormal columns is an orthogonal matrix.

e.) If L is a line through $\vec{0}$ and if $\hat{y}$ is the orthogonal projection of $\vec{y}$ onto L, then $\|\hat{y}\|$ gives the distance from $\hat{y}$ to L.

a.) True b.) True c.) False d.) False e.) False

26.) Suppose W is a subspace of $\mathbb{R}^n$ spanned by n nonzero orthogonal vectors. Explain why $W = \mathbb{R}^n$.

A set of nonzero orthogonal vectors in $\mathbb{R}^n$ is a linearly indep. set by Thm 4. Since W is spanned by these vectors, we have that the set is a basis for W. Thus $\dim W = n$. The only n-dimensional subspace of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself, so $W = \mathbb{R}^n$.

27.) Let U be a square matrix with orthonormal columns. Explain why U is invertible.

If the columns of U are orthonormal, they are orthogonal and any orthogonal subset of $\mathbb{R}^n$ is linearly independent (Thm 4). Thus the columns of U are linearly independent, so U is invertible by the IMT.

28.) Let U be an $n \times n$ orthogonal matrix. Show that the rows of U form an orthonormal basis of $\mathbb{R}^n$.

An orthogonal matrix is a square, invertible matrix U such that $U^{-1} = U^T$. Since U is invertible, $I = UU^{-1} = UU^T = (U^T)^T U^T$. Since $(U^T)^T U^T = I$, by thm 6, U^T has orthonormal columns, which means U has orthonormal rows.

Since U^T is invertible, by the IMT, its columns span $\mathbb{R}^n$. Thus the rows of U span $\mathbb{R}^n$. Hence the rows of U are an orthonormal basis for $\mathbb{R}^n$.

29.) Let U and V be $n \times n$ orthogonal matrices. Explain why UV is an orthogonal matrix.

U and V are invertible, so UV is invertible and $(UV)^{-1} = V^{-1}U^{-1}$.

Also $U^{-1} = U^T$ and $V^{-1} = V^T$ so $(UV)^{-1} = V^{-1}U^{-1} = V^T U^T = (UV)^T$.

Since UV is invertible and $(UV)^{-1} = (UV)^T$, UV is an orthogonal matrix.

6.3 # 1, 6, 7, 9, 11, 13, 17, 21, 23, 24

1.) Assume $\{\vec{u}_1, \dots, \vec{u}_4\}$ is an orthogonal basis for $\mathbb{R}^4$.

$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}$, $\vec{u}_4 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix}$ Write $\vec{x}$ as a sum of two vectors, one in Span$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, the other in Span$\{\vec{u}_4\}$.

The vector in Span$\{\vec{u}_4\}$ is $\dfrac{\vec{x} \cdot \vec{u}_4}{\vec{u}_4 \cdot \vec{u}_4} u_4 = \dfrac{50+24-2}{25+9+2} \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$

Since $\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$, we can solve for $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$.

$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix}$ is in Span$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$

So $\vec{x} = \begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$ where $\begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix} \in$ Span$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and $\begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix} \in$ Span$\{\vec{u}_4\}$.

6.) Verify that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set, and then find the orthogonal Projection of $\vec{y}$ onto Span$\{\vec{u}_1, \vec{u}_2\}$.

$\vec{y} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\vec{u}_1 \cdot \vec{u}_2 = -4(0) + -1(1) + 1(1) = 0$ So $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set.

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \frac{y \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2 = \frac{-24-4+1}{16+1+1}\begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{0+4+1}{0+1+1}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = -\frac{3}{2}\begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

$$\boxed{\hat{y} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}}$$

7.) Let W be the subspace spanned by the $\vec{u}$'s and write $\vec{y}$ as a sum of a vector in W and a vector orthogonal to W.

$\vec{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$

$\vec{u}_1 \cdot \vec{u}_2 = 5 + 3 - 8 = 0$ So $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set. Since W is also spanned by the $\vec{u}$'s, $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis of W.

We then use the Orthogonal Decomposition Theorem. $\vec{y} = \hat{y} + \vec{z}$ where $\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2$ is in W and $\vec{z} = \vec{y} - \hat{y}$ is in $W^\perp$.

$$\hat{y} = \frac{1+9-10}{1+9+4} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{5+3+20}{25+1+16} \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = 0 + \frac{2}{3} \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} \qquad \vec{z} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} = \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}.$$

$$\boxed{\vec{y} = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} + \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}}$$

9.) $\vec{y} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$\vec{u}_1 \cdot \vec{u}_2 = -1 + 3 + 0 - 2 = 0$

$\vec{u}_1 \cdot \vec{u}_3 = -1 + 0 + 0 + 1 = 0$

$\vec{u}_2 \cdot \vec{u}_3 = 1 + 0 + 1 - 2 = 0$

$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set that spans W, so it is an orthogonal basis of W. We again use the Orthogonal Decomposition Theorem.

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 + \frac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} \vec{u}_3 = \frac{4+3-1}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \frac{-4+9+3+2}{1+9+1+4} \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix} + \frac{-4+3-1}{3} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -2/3 \\ 2 \\ 2/3 \\ -4/3 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 0 \\ -2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix} \qquad \boxed{\vec{y} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix}}$$

11.) Find the closest point to $\vec{y}$ in the subspace W spanned by $\vec{v}_1, \vec{v}_2$.

$\vec{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_2 = 3 - 1 - 1 - 1 = 0$, So $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal set that spans W, so it is an orthogonal basis.

By the Best approximation theorem, $\hat{y}$ is the closest point in W to $\vec{y}$.

$$\hat{y} = \frac{9+1-5+1}{9+3} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{3-1+5-1}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$\boxed{\hat{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix} \text{ is the closest Point in } W \text{ to } \vec{y}}$

6.3 continued

13.) Find the best approximation to $\vec{z}$ by vectors of the form $c_1\vec{v}_1 + c_2\vec{v}_2$.

$\vec{z} = \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_2 = 2-1-1=0$ So $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal set. The set of vectors of the form $c_1\vec{v}_1 + c_2\vec{v}_2$ is the Span$\{\vec{v}_1, \vec{v}_2\}$. $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for Span$\{\vec{v}_1, \vec{v}_2\}$ and the best approximation for $\vec{z}$ in this space is

$$\hat{z} = \frac{\vec{z}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{z}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 = \frac{6+7-6+3}{4+2+9}\begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix} + \frac{3-7-3}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix} - \frac{7}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -2 \\ 3 \end{bmatrix}$$

17.) Let $\vec{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$ and $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$.

a.) Let $U = [\vec{u}_1 \ \vec{u}_2]$. Compute U^TU and UU^T.

b.) Compute $\text{proj}_W \vec{y}$ and $(UU^T)\vec{y}$.

a.) $$U^TU = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 4/9+1/9+4/9 & -4/9+2/9+4/9 \\ -4/9+2/9+2/9 & 4/9+4/9+1/9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$UU^T = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 4/9+4/9 & 2/9-4/9 & 4/9-2/9 \\ 2/9-4/9 & 1/9+4/9 & 2/9+2/9 \\ 4/9-2/9 & 2/9+2/9 & 4/9+1/9 \end{bmatrix} = \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix}$$

b.) $\vec{u}_1 \cdot \vec{u}_2 = -4/9 + 2/9 + 2/9 = 0$
$\vec{u}_1 \cdot \vec{u}_1 = 4/9 + 1/9 + 4/9 = 1$
$\vec{u}_2 \cdot \vec{u}_2 = 4/9 + 4/9 + 1/9 = 1$

$\{\vec{u}_1, \vec{u}_2\}$ is an orthonormal basis for W, so by thm 10 (pg 351 Lay) $\text{proj}_W \vec{y} = (UU^T)\vec{y}$

$$\text{Proj}_W \vec{y} = (UU^T)\vec{y} = \frac{1}{9}\begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 32-16+2 \\ -8+40+4 \\ 8+32+5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

21.) True / False. All vectors and subspaces are in $\mathbb{R}^n$.

a.) If $\vec{z}$ is orthogonal to $\vec{u}_1$ and to $\vec{u}_2$ and if $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$ then $\vec{z}$ must be in $W^{\perp}$.

b.) For each $\vec{y}$ and each subspace W, the vector $\vec{y} - \text{proj}_W \vec{y}$ is orthogonal to W.

c.) The orthogonal projection $\hat{y}$ of $\vec{y}$ onto a subspace W can sometimes depend on the orthogonal basis for W used to compute $\hat{y}$.

d.) If $\vec{y}$ is in a subspace W, then the orthogonal projection of $\vec{y}$ onto W is $\vec{y}$ itself.

e.) If the columns of an $n \times p$ matrix U are orthonormal, then $UU^T\vec{y}$ is the orthogonal projection of $\vec{y}$ onto the column space U.

a.) TRUE b.) TRUE c.) FALSE d.) TRUE e.) TRUE

23.) Let A be an $m \times n$ matrix. Prove that every vector $\vec{x}$ in $\mathbb{R}^n$ can be written in the form $\vec{x} = \vec{p} + \vec{u}$, where $\vec{p}$ is in $\text{Row} A$ and $\vec{u}$ is in $\text{Nul} A$. Also, show that if the equation $A\vec{x} = \vec{b}$ is consistent, then there is a unique $\vec{p}$ in $\text{Row} A$ s.t. $A\vec{p} = \vec{b}$.

By the orthogonal decomposition thm any $\vec{x} \in \mathbb{R}^n$ can be written as $\vec{x} = \vec{p} + \vec{u}$ where $\vec{p} \in \text{Row} A$ and $\vec{u} \in (\text{Row} A)^{\perp}$. By thm 3 in Lay 6.1, $(\text{Row} A)^{\perp} = \text{Nul} A$.

Next, if $A\vec{x} = \vec{b}$ is consistent, then let $\vec{x}$ be a solution and $\vec{x} = \vec{p} + \vec{u}$ where $\vec{p} \in \text{Row} A$

Then $A\vec{p} = A(\vec{x} - \vec{u}) = A\vec{x} - A\vec{u} = \vec{b} - \vec{0} = \vec{b}$. (Since $\vec{u} \in \text{Nul} A$, $A\vec{u} = \vec{0}$) Thus $A\vec{p} = \vec{b}$ has at least one soln. Suppose there is more than one soln, that is, for $\vec{p} \neq \vec{p}'$ $A\vec{p} = \vec{b}$ and $A\vec{p}' = \vec{b}$. Then $A\vec{p} = A\vec{p}'$ implies $A\vec{p} - A\vec{p}' = A(\vec{p} - \vec{p}') = \vec{0}$ then $\vec{p} - \vec{p}' \in \text{Nul} A = (\text{Row} A)^{\perp}$.

$\vec{p} = \vec{p}' + (\vec{p} - \vec{p}')$ satisfies the orthogonal decomposition theorem, but so does $\vec{p} = \vec{p} + \vec{0}$. Since the orthogonal decomposition theorem gives a unique vector, $\vec{p}' = \vec{p}$. Therefore $\vec{p}$ is unique.

6.3 Continued

24.) Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\vec{w}_1, \dots \vec{w}_p\}$ and let $\{\vec{v}_1, \dots \vec{v}_q\}$ be an orthogonal basis for $W^\perp$.

a.) Explain why $\{\vec{w}_1, \dots \vec{w}_p, \vec{v}_1, \dots \vec{v}_q\}$ is an orthogonal set.

Since $\{\vec{w}_1, \dots, \vec{w}_p\}$ is an orthogonal basis, by definition these vectors are pairwise orthogonal. Similarly for $\{\vec{v}_1, \dots \vec{v}_q\}$. Then for any $\vec{v}_j \in \{\vec{v}_1, \dots \vec{v}_q\}$ and any $\vec{w}_i$ in $\{\vec{w}_1, \dots, \vec{w}_p\}$, $\vec{w}_i \in W$ and $\vec{v}_j \in W^\perp$, so $\vec{w}_i \cdot \vec{v}_j = \vec{0}$. Therefore $\{\vec{w}_1, \dots \vec{w}_p, \vec{v}_1, \dots, \vec{v}_q\}$ is orthogonal.

b.) Explain why the set in part (a) spans $\mathbb{R}^n$.

For any $\vec{y} \in \mathbb{R}^n$, $\vec{y} = \hat{y} + \vec{z}$ where $\hat{y} \in W$ and $\vec{z} \in W^\perp$ by the orthogonal decomposistion theorem. Since $\hat{y} \in W$, $\hat{y} = a_1\vec{w}_1 + \dots + a_p\vec{w}_p$ for some scalars a_i since $\{\vec{w}_1, \dots, \vec{w}_p\}$ is a basis for W. Similarly $\vec{z} = b_1\vec{v}_1 + \dots + b_q\vec{v}_q$, for some scalars b_j. Thus $\vec{y} = a_1\vec{w}_1 + \dots + a_p\vec{w}_p + b_1\vec{v}_1 + \dots b_q\vec{v}_q$. Thus any $\vec{y} \in \mathbb{R}^n$ can be written as a linear combination of $\{\vec{w}_1, \dots \vec{w}_p, \vec{v}_1, \dots \vec{v}_q\}$. So this set spans $\mathbb{R}^n$.

c.) Show that $\dim W + \dim W^\perp = n$.

Part (a) tells us $\{\vec{w}_1, \dots \vec{w}_p, \vec{v}_1, \dots \vec{v}_q\}$ is linearly independent, and part (b) tells us it spans $\mathbb{R}^n$. Therefore this set is a basis for $\mathbb{R}^n$.

$\dim \mathbb{R}^n = p + q$, but we know $\dim \mathbb{R}^n = n$ and $\dim W = p$ since $\{\vec{w}_1, \dots, \vec{w}_p\}$ is a basis and $\dim W^\perp = q$ since $\{\vec{v}_1, \dots, \vec{v}_q\}$ is a basis.

Hence $\quad n = \dim W + \dim W^\perp$