University of Connecticut
Department of Mathematics

MATH 2210 CHAPTER 5 AND 6 PRACTICE PROBLEMS FALL 2018

NAME: ______________________________

Instructor Name: ______________________________ Section: __________

This practice exam ***is not*** an exhaustive list of problems you should expect to see on the actual exam. Also, it's ***way*** longer than the in class exam. Look at the homework, quizzes, in class notes, or extra suggested problems from the book for more practice.

The following message appears on the actual exam:

Read This First!

- Please read each question carefully. Show **ALL** work clearly in the space provided. In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- Answers must be clearly labeled in the spaces provided after each question. Please cross out or fully erase any work that you do not want graded. The point value of each question is indicated after its statement. No books or other references are permitted.
- No calculators permitted.

Grading - For Administrative Use Only

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1 Chapter Summaries

This list is incomplete. Here are some of the important topics covered in each chapter on the exam.

Chapter 5

5.1 Eigenvectors and Eigenvalues
- how to determine if a scalar is an eigenvalue
- how to determine if a vector is an eigenvector
- how to find eigenspace given an eigenvalue
- how to find eigenvalues of a triangular matrix
- row operations change eigenvalues
- what does it mean to have an eigenvalue of zero

5.2 The Characteristic Equation
- how to find eigenvalues and their multiplicities using the characteristic equation

5.3 Diagonalization
- know how to use eigenvalues and eigenvectors to diagonalize a matrix
- definition of eigenvector basis

5.4 Eigenvectors and Linear Transformations
- definition of matrix for T relative to $\mathcal{B}$ and $\mathcal{C}$
- definition of $\mathcal{B}$-matrix for a transformation $T : V \to V$
- diagonalizable matrix transformations
- take home exam, video, and quiz 10

1 **Chapter 6**

6.1 Inner Product, Length, Orthogonality
- dot product
- norm
- distance
- orthogonality

6.2 Orthogonal Sets
- orthogonal set
- orthogonal basis
- matrices with orthonormal columns, and orthogonal matrices

6.3 Orthogonal Projections
- orthogonal decomposition theorem
- best approximation theorem
- orthonormal bases

6.4 The Gram-Schmidt Process
- how to find an orthogonal/orthonormal basis
- how to find a QR factorization

2 Practice Exam

1. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{bmatrix}$.

   (a) Find an eigenvector basis for $\mathbb{R}^3$ corresponding to A.

Find eigenvalues

$$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 1 & 1 \\ 2 & 1-\lambda & 2 \\ 3 & 3 & 2-\lambda \end{pmatrix} =$$

$$= -\lambda\big((1-\lambda)(2-\lambda) - 6\big) - \big(2(2-\lambda)-6\big) + \big(6 - 3(1-\lambda)\big)$$

$$= -\lambda^3 + 3\lambda + 9\lambda + 5 = -(\lambda+1)^2(\lambda-5)$$

$$\Rightarrow \lambda = -1, 5$$

$$E_{\lambda=5} = \mathrm{Null}\left(\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix}\right) = \mathrm{Span}\left\{\begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}\right\}$$

$$E_{\lambda=-1} = \mathrm{Null}\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}\right) = \mathrm{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

   (b) Diagonalize A.

$$P = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

2. Let $T : \mathbb{P}^2 \to \mathbb{P}^2$ be defined by

$$T(a_0 + a_1 t + a_2 t^2) = 3a_0 + (5a_0 - 2a_1)t + (4a_1 + a_2)t^2.$$

(a) Show that T is linear.

$$T\big((a_0+a_1t+a_2t^2)+(b_0+b_1t+b_2t^2)\big)$$
$$T\big((a_0+b_0)+(a_1+b_1)t+(a_2+b_2)t^2\big)$$
$$= 3(a_0+b_0)+5\big((a_0+b_0)-2(a_1+b_1)t\big)+\big(4(a_1+b_1)+(a_2+b_2)\big)t^2$$
$$= \big(3a_0+(5a_0-2a_1)t+4(4a_1+a_2)t^2\big)+\big(3a_0+(5a_0-2a_1)t+4(4a_1+a_2)t^2\big)$$
$$= T(p)+T(q)$$

also need to check if $T(cp) = cT(p)$

(b) Find the matrix representation for T relative to the standard basis for $\mathbb{P}^n$.

$$M = \big[\,[T(b_0)]_B \;\; [T(b_1)]_B \;\; [T(b_2)]_B\,\big]$$

$$T(b_0) = T(1) = 3+5t \implies [T(b_0)]_B = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$$
$$T(b_1) = T(t) = -2t+4t^2 \implies [T(b_1)]_B = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$$
$$T(b_2) = T(t^2) = t^2 \implies [T(b_2)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \implies M = \begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

(c) Find the kernel and range of T using part (b).

let $S : \mathbb{R}^3 \to \mathbb{R}^3$ by $\vec{x} \mapsto M\vec{x}$

S has kernel $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ← (b/c M invertible)

S has range $\mathbb{R}^3$

$\implies T$ has kernel $\{\vec{0}\}$

T has range $\mathbb{P}_2$

3. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}$.

(a) Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

$$\begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix} = -14 + 20 - 6 = 0$$

should be squares

(b) Verify the Pythagorean theorem by checking $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$.

$$\|u\| = \left\|\begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}\right\|^2 = 4 + 25 + 1 = 30$$

$$\|v\| = \left\|\begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}\right\|^2 = 49 + 16 + 36 = 101$$

$$\|u + v\| = \left\|\begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} + \begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}\right\|^2 = \left\|\begin{bmatrix} -5 \\ -9 \\ 5 \end{bmatrix}\right\|^2$$

$$= 25 + 81 + 25 = 131$$

4. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 8 \\ -4 \\ 3 \end{bmatrix}$.

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an ~~orthonormal~~ orthogonal basis for $\mathbb{R}^3$.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

$$\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = -2 + 4 - 2 = 0$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 2 + 0 - 2 = 0$$

$\Rightarrow$ all orthogonal

(b) Express $\mathbf{x}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

$$\vec{x} = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{x \cdot u_3}{u_3 \cdot u_3} u_3$$

$$= \frac{11}{2} u_1 - \frac{7}{6} u_2 + \frac{2}{3} u_3$$

5. Find the distance from $\mathbf{y}$ to $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \text{ and } \mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

HAVE TO CHECK $\{v_1, v_2\}$ orthogonal first

$$v_1 \cdot v_2 = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} = 9 + 1 - 5 + 1 = \underline{6}$$

it's $\underline{\text{not}}$

find orthogonal basis

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}} \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 15/6 \\ 5/6 \\ -11/6 \\ 5/6 \end{bmatrix} \qquad v_2' = \begin{bmatrix} 15 \\ 5 \\ -11 \\ 5 \end{bmatrix}$$

$$\hat{y} = \frac{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}} \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 15 \\ 5 \\ -11 \\ 5 \end{bmatrix}}{\begin{bmatrix} 15 \\ 5 \\ -11 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 15 \\ 5 \\ -11 \\ 5 \end{bmatrix}} \begin{bmatrix} 15 \\ 5 \\ -11 \\ 5 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} + \frac{1}{66} \begin{bmatrix} 15 \\ 5 \\ -11 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3/11 \\ 1/11 \\ 1 \\ 1/11 \end{bmatrix} \Longrightarrow \|y - \hat{y}\| = \left\| \begin{bmatrix} 8/11 \\ -12/11 \\ 0 \\ -12/11 \end{bmatrix} \right\|$$

$$= 4\sqrt{\frac{2}{11}}$$

6. Find an orthonormal basis for the column space of

$$\begin{bmatrix} 3 & 8 \\ 0 & 5 \\ -1 & -6 \end{bmatrix}.$$

$$\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} = 24 + 0 + 6 = 30$$

__not__ orthogonal

$$v_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} - \frac{\begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$$

$$\|v_1\| = \sqrt{10}$$
$$\|v_2\| = \sqrt{35} \Rightarrow u_1 = \begin{bmatrix} 3/\sqrt{10} \\ 0 \\ -1/\sqrt{10} \end{bmatrix} \quad u_2 = \begin{bmatrix} -1/\sqrt{35} \\ 5/\sqrt{35} \\ -3/\sqrt{35} \end{bmatrix}$$

7. Find an orthogonal basis for the column space of

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}.$$

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \frac{\begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}}{\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\mathcal{B} = \{v_1, v_2, v_3\}$$