For questions 1-4, read the given information and decide if each statement is True or False. Circle your answer. You do not need to justify your answer. No partial credit will be given.

1. Consider the set $S = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \right\}$. [3]

   (a) S is a basis for $\mathbf{R}^3$. True False
   (b) S is an orthogonal set. True False
   (c) The vectors in S are normal. True False

2. Consider the matrix $A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$. [4]

   (a) The equation $A\vec{x} = \vec{b}$ is consistent for any $\vec{b} \in \mathbf{R}^2$. True False
   (b) The equation $A\vec{x} = \vec{0}$ has exactly one solution, the trivial solution. True False
   (c) A is invertible. True False
   (d) The equation $A\vec{x} = \vec{0}$ has infinitely many solutions, all of which are non-trivial. True False

   weird wording. Technically true.

3. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \end{bmatrix} \right\}$ be a basis for $\mathbf{R}^2$ and let $\vec{x} = -1\vec{b}_1 + 2\vec{b}_2$. [4]

   (a) $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. True False
   (b) $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -7 \\ -11 \end{bmatrix}$. True False
   (c) $P_{\mathcal{B}} = \begin{bmatrix} 5 & -1 \\ 3 & -4 \end{bmatrix}$. True False
   (d) $P_{\mathcal{B}}$ sends $\vec{x}$ to $[\vec{x}]_{\mathcal{B}}$. True False

4. Let W be a vector space and let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ be a basis for some subspace V of W. [5]

   (a) The set $\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly independent. True False
   (b) For any $\vec{x} \in V$, $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{x}\} = V$. True False
   (c) For any $\vec{x} \in V$, the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{x}\}$ is linearly independent. True False
   (d) The set $\{\vec{v}_1, \vec{v}_2\}$ spans V. True False
   (e) $\dim V \leq \dim W$. True False

   could be =

5. Answer the following using the matrix below.

$$A = \begin{bmatrix} 1 & -6 \\ -6 & 10 \end{bmatrix}$$

(a) Compute $\det A$. [2]

$1 \cdot 10 - (-6 \cdot -6) = 10 - 36 = -26$

(b) Compute A^{-1}. [2]

$\frac{1}{-26}\begin{bmatrix} 10 & 6 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} -5/13 & -3/13 \\ -3/13 & -1/26 \end{bmatrix}$

(c) Find the eigenvalues of A. [2]

$\lambda = -2, 13$

$\det\begin{bmatrix} 1-\lambda & -6 \\ -6 & 10-\lambda \end{bmatrix} = (1-\lambda)(10-\lambda) - 36$

$= 10 + \lambda^2 - 11\lambda - 36$

$= \lambda^2 - 11\lambda - 26 = (\lambda - 13)(\lambda + 2)$

(d) Is A diagonalizable? Explain why or why not. If so, you do not need to find the factorization. [2]

yes, distinct eigen values

6. Answer the following using the matrix below.

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & -2 & 2 \\ 3 & 0 & 6 \end{bmatrix}$$

(a) Compute $\det A$. [4]

$$\det \begin{bmatrix} 1 & 1 & 4 \\ 2 & -2 & 2 \\ 3 & 0 & 6 \end{bmatrix} = 3 \cdot \begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix} + 6 \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix}$$

$$= 3(2+8) + 6(-2-2)$$

$$= 30 + (-24) = 6$$

(b) Compute A^{-1}. [4]

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 2 & -2 & 2 & 0 & 1 & 0 \\ 3 & 0 & 6 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 5/3 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1/2 & -2/3 \end{array}\right]$$

7. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$\frac{8}{30} = \frac{4}{15}$

(a) Use the Gram-Schmidt process to construct and orthogonal basis for Col A. [7]

$\frac{2}{10} \quad \frac{1}{5}$

$v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$v_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} - \frac{2}{6} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 5/3 \\ 1/3 \end{bmatrix}$

let $v_2 = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$

$v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \cancel{\frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}} - \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}}{\begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}} \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{30} \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 0 \\ 4/5 \end{bmatrix}$

let $v_3 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$

then $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \right\}$

(b) Find a basis for each eigenspace of A given that the eigenvalues of A are $\lambda = 1$ and $\lambda = 2$. [7]

$E_1 = \text{Null}\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} \right) = \text{Null}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right)$

$= \text{Span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

$E_2 = \text{Null}\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix} \right) = \text{Null}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$

$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad = \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(c) Find the P and D used to diagonalize A. You do not need to find P^{-1}. [2]

$P = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

8. Use Cramer's Rule to find a solution to $A\vec{x} = \vec{b}$ where A is the **triangular** matrix below and $\vec{b}$ is the vector below. Show all work, no credit will be awarded for any other method. [8]

$$A = \begin{bmatrix} -1 & 4 & 3 \\ 0 & -3 & 7 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$A_1(b) = \begin{bmatrix} 4 & 4 & 3 \\ 0 & -3 & 7 \\ 0 & 0 & 1 \end{bmatrix} \quad \det(A_1(b)) = -12$$

$$A_2(b) = \begin{bmatrix} -1 & 4 & 3 \\ 0 & 0 & 7 \\ 0 & 0 & 1 \end{bmatrix} \quad \det(A_2(b)) = 0$$

$$A_3(b) = \begin{bmatrix} -1 & 4 & 4 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \det(A_3(b)) = 0$$

$$A = \begin{bmatrix} -1 & 4 & 3 \\ 0 & -3 & 7 \\ 0 & 0 & 1 \end{bmatrix} \quad \det(A) = 3$$

$x_1 = -4 \qquad x_2 = 0 \qquad x_3 = 0$

9. Check your answer to the previous question. (If you were unable to answer the question, take a guess for the solution and show if your guess is correct or incorrect.) correct incorrect [2]

$$\begin{bmatrix} -1 & 4 & 3 \\ 0 & -3 & 7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

10. Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be given by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$.

(a) Show that $T(\vec{x})$ is a linear transformation. [6]

$$T(k\vec{u}+\vec{v}) = A(k\vec{u}+\vec{v})$$
$$= A(k\vec{u}) + A\vec{v}$$
$$= kA\vec{u} + A\vec{v} = kT(\vec{u}) + T(\vec{v})$$

✓

(b) Is T one-to-one? Explain why or why not. [3]

no, the columns are linearly dep

(c) Find a vector in the range of T. Explain how you know this vector is in the range of T. [3]

$\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ b/c $\mathrm{Col}(A) = \mathrm{Range}(T)$

11. Let $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$ and let A be the matrix whose columns are $\vec{u}_1$ and $\vec{u}_2$.

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(a) Find the vector in W closest to $\vec{b}$. [3]

$$\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \neq 0$$

$$u_2' = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}}{\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{8} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \cancel{\frac{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}}{\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}} - \frac{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

[3]

[3]

12. Construct a 3×4 matrix A that is in RREF (row reduced echelon form). Construct A such that it has linearly **dependent** columns and $A\vec{x} = \vec{b}$ is not consistent for every $\vec{b} \in \mathbf{R}^3$. [3]

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

lots of answers

13. (a) Let $W = \left\{ \begin{bmatrix} 2a+b \\ c-b \end{bmatrix} : a, b, c \in \mathbf{R} \right\}$. Prove or disprove that W is a subspace of $\mathbf{R}^2$. [6]

$= \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow$ subspace

(b) Write W as a span of vectors. i.e. Find vectors $\vec{v}_1, \ldots, \vec{v}_n$ such that $W = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$. [2]

$= \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

14. Let $\mathcal{B} = \{1+t+t^2, -t-t^2, t\}$ be a basis for $\mathbf{P}_2$. Find the coordinate vector for the polynomial $p(t) = 5t^2 + 6t - 1$ with respect to $\mathcal{B}$. [5]

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} [x]_B = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$$

$$\Longleftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 6 \\ 1 & -1 & 0 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$[\vec{p}]_B = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$$

Bonus: Write as many equivalences of the invertible matrix theorem as you can. Let A be an $n \times n$ matrix. Then A is invertible if and only if: