Fall 2019: Classical Field Theory (PH6297)

Special Relativity reloaded

August 7, 2019

1 Introduction: Lab frame and the boosted frame

We always keep in mind our two friends, the inertial frame, S who is at rest wrt us, which we will call the "Lab frame". And then we have the other frame, S', which is "boosted" wrt to the lab frame by a constant velocity, $\mathbf{v}$. In a typical physical situation, the lab frame could be the experimenter's frame while the boosted frame could be the frame attached to a (uniform) moving object such as an elementary particle, say the electron or a fluid streamline. Since the boosted frame or the electron/fluid frame is stuck on to the object itself, it is the "Rest frame" of the electron (the electron doesn't move wrt to itself).

An event P is labeled by in a frame by the coordinates where and when it happened. For example in lab frame, P occurs at $x^\mu = (x^0, x^1, x^2, x^3)$. Here x^0 denotes the time at which P happened, $x^0 = c\,t$[1] while (x^1, x^2, x^3) are the spatial coordinates giving the location where P happened, $(x^1, x^2, x^3) = (x, y, z) = \mathbf{x}$. Now the observer in the boosted frame, S' will of course disagree and instead claim that P happened somewhere else and sometime else, denoted by $x'^\mu = (x'^0, x'^1, x'^2, x'^3)$. Lorentz transformation tells us how to go back and forth between the two frames/observers, i.e. it tells us how to take the coordinates of an event in$S$ and convert it into coordinates of the same event in boosted frame S' and vice verse. This conversion map, i.e. the Lorentz transformation map (Λ) will obviously depend on how much S' is boosted with respect to S, i.e. on the "boost parameter", $\mathbf{v}$, i.e.

$$\Lambda = \Lambda(\mathbf{v}).$$

Before we move on we introduce two *dimensionless* quantities which will clean up notations from now on. Introduce,

$$\begin{aligned} \boldsymbol{\beta} &= \mathbf{v}/c, \\ \gamma &= \frac{1}{\sqrt{1-\boldsymbol{\beta}^2}} = \frac{1}{\sqrt{1-\mathbf{v}^2/c^2}}. \end{aligned}$$

$\boldsymbol{\beta}$ is called the boost parameter and γ is called the Lorentz factor. Since in special relativity no material object can move with a speed more than the speed of light, we have

$$\begin{aligned} |\boldsymbol{\beta}| &< 1, \\ \gamma &> 1. \end{aligned}$$

[1] c is the maximal signal speed (i.e. speed of light in vacuo). We will work on units where the speed of light is unity, $c = 1$. Note that not only speed of light is unity but in these units it is dimensionless, i.e. length and time have same dimensions.

Using these we can rewrite the Lorentz transformation rule in a cleaner way for the special case when the boost is along the x-axis, i.e. when $\boldsymbol{\beta} = \beta\,\hat{\boldsymbol{x}}$,

$$\begin{aligned} x'^1 &= \gamma\, x^1 - \gamma\beta\, x^0, \\ x'^0 &= \gamma\, x^0 - \gamma\beta\, x^1, \\ x'^2 &= x^2, \\ x'^3 &= x^3. \end{aligned}$$

Since these are a bunch of linear equations, one write all four equations as an equation involving a single 4×4 matrix:

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & & \\ & & & \end{pmatrix}}\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{1}$$

$$x' = \Lambda(\beta\hat{\mathbf{x}}).\quad x. \tag{2}$$

In the last line we have denoted the column *4-vector* $x' = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}$ and the Lorenz boost matrix, $\Lambda\left(\beta\hat{\mathbf{x}}\right) = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & & \\ & & & \end{pmatrix}$.

2 A Compendium of results on 4-vector component notation

- For a general case of Lorentz transformations (i.e. boosts or spatial rotations), the Lorentz transformations look like
$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu.$$
Of course this secretly assumes that at time $t = 0$, the origins of the two frames were coincident. In the general case one could have the origins to be separated at $t = 0$ and the clocks to be not synchronized (we label these *constant* space and time shifts as *translations* and denote them by a^μ). In the most general case, we would have[2],
$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu + a^\mu. \tag{3}$$

[2]This general case which includes all of boosts, rotations, and translations is called the a *Poincaré* transformation. $\Lambda^0{}_i$) represent boosts; $\Lambda^i\, j$ represent rotations and a^μ's represent translations. This 10 transformation group (i.e. 3 boosts, 3 rotations and 4 translations) is called the *Poincaré* group. All elementary particles are irreps (irreducible representations) of the Poincaré group.

So it's better to work to coordinate differences/differentials so the shifts drop out and we have[3] ,

$$
\begin{aligned}
dx'^{\mu} &= \Lambda^{\mu}{}_{\nu}\, dx^{\nu}, &(4)\\
\Longrightarrow \frac{\partial x'^{\mu}}{\partial x^{\nu}} &= \Lambda^{\mu}{}_{\nu}. &(5)
\end{aligned}
$$

- **The Invariant (squared) spacetime interval**: Let's consider a pair of events occurring at spacetine location x and $x+dx$ according to the lab frame while the same pair of events appear to occur at x' and $x'+dx'$ in the rest frame. In special relativity the time interval of two nearby events is **not same** in all inertial frames, and neither is the spatial separation i.e. $dt' \neq dt$ and $|d\mathbf{x}'| \neq |d\mathbf{x}|$[4]. But, one check that under a boost (1), the squared spacetime interval remains unchanged

$$ds^2 = c^2dt^2 - d\mathbf{x}\cdot d\mathbf{x} = c^2dt'^2 - d\mathbf{x}'\cdot d\mathbf{x}'.$$

One can easily check this using for a boost along x-axis, (1). In fact, one can actually check that even for *arbitrary* Lorentz transformation this is true,

$$ds^2 = \left(dx^0\right)^2 - d\mathbf{x}\cdot d\mathbf{x} = \left(dx'^0\right)^2 - d\mathbf{x}'\cdot d\mathbf{x}'. \tag{6}$$

In a sense ds^2 captures how far in both space and time two events occur and whether there is any causal connection between them i.e. if is it possible for them to be connected as cause and effect. If for two nearby events, $ds^2 > 0$, the two events are said to be *timelike separated* and in fact it is possible a signal to travel from one event to the other and thus they can be connected as cause and effect. If $ds^2 = 0$, then the two events are said to be *lightlike separated*, and they can be joined by a light signal traveling from either one event to the other. However, if $ds^2 < 0$, the two events are said to be *spacelike separated* and no signal can be sent from one event to the other, which means they cannot be connected by cause and effect.

W can the squared interval in matrix form which is coordinate/index free,

$$ds^2 = \begin{pmatrix} dx^0 & dx^1 & dx^2 & dx^3 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} = dx^T\, \eta\, dx.$$

where we have introduced Minkowski metric as the matrix η,

$$\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix},$$

[3]Note that when we go over to GR, Lorentz transformations (LT) would generalize to general coordinate transformations (GCT), BUT this formula will still hold except the $\Lambda^{\mu}{}_{\nu}$ will not be constants but functions of spacetime,

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = [\Lambda(x)]^{\mu}{}_{\nu}.$$

[4]In Galilean relativity, one *would* have $dt' = dt$ (Newton's notion of absolute time) and $|d\mathbf{x}'| \neq |d\mathbf{x}|$.

which is a symmetric matrix, $\eta = \eta^T$ and in particular diagonal. Note that it is self-inverse, $\eta^2 = 1$.

- **The Lorentz Group:** To determine what is the nature of the Lorentz transformation matrices, Λ's, we start by equating the expression for the invariant squared interval in the two frames,

$$\begin{aligned} ds^2 = dx^T\, \eta\, dx &= dx'^T\, \eta\, dx'. \\ &= (\Lambda\, dx)^T\; \eta\; (\Lambda\, x) \\ &= dx^T\; \left(\Lambda^T \eta \Lambda\right)\; dx. \end{aligned}$$

This implies,

$$\Lambda^T\, \eta\, \Lambda = \eta. \tag{7}$$

If in place of η one had the identity matrix, it would have meant Λ is an orthogonal matrix. But the $\eta_{00} = -1$ spoils this. Instead we call Λ an $O(1,3)$ matrix (O for orthogonal and the$(1,3)$ refers to the metric signature i.e. number of positive and negative eigenvalues respectively of η which is $-+++$).
Taking determinants of both sides, we get

$$(\det\Lambda)^2 = 1 \implies \det\Lambda = \pm 1.$$

Since the Lorentz transformation is continuously connected to unity (i.e. no transformation), we will stick to $\det\Lambda = +1$. $O(1,3)$ matrices which are unit determinant are called "Special orthogonal" and we denote them by putting an extra "S" in front: $SO(1,3)$[5]. The fact that Lorentz transformation matrices are not orthogonal will have an important consequence as we will see momentarily, namely, one will have to introduce two different vector species in special relativity, one with upper indices (superscripts) and the other with lower indices (subscripts). It will also turn out that this is the precise reason we had to denote the Lorentz transformation with one up and one down index. Thus coordinate differentials, dx^μ and coordinate partials ∂_μ do not transform identically under a Lorentz transformation despite the fact that they both have one index, namely μ.
(This is unlike the case of Galilean transformation matrices, O_{ij}, as they being rotation matrices orthogonal matrices, $O = O^{-T}$ and hence coordinate differentials dx^i and partial derivatives ∂_i transform identically)

- **Transformation of coordinate partials:** We already know how coordinate differentials, dx^μ transform under Lorentz transformation, namely Eq. (4). Here we are curious to know how do the coordinate partial derivative operators $\frac{\partial}{\partial x^\mu}$ transform after a Lorentz transformation? We will use a condensed notation denoting,

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}.$$

Let's say it is some transformation rule like this,

$$\frac{\partial}{\partial x'^\rho} = \bar{\Lambda}_\rho{}^\sigma \frac{\partial}{\partial x^\sigma},$$

[5] One can generalize this to some different η which has p number of diagonal elements which are -1 and q number of diagonal elements which are +1. Then we would call the transformation matrix, Λ to be an $SO(p,q)$ type matrix. This is the relevant for example in the context of **Anti de sitter (AdS) space** (we need$SO(2,3)$) which is of great relevance in current day string theory research.

where we are yet to determine $\bar{\Lambda}_\mu{}^\nu$. This can be easily determined by acting both sides on x'^μ,

$$\begin{aligned}
\frac{\partial}{\partial x'^\rho} x'^\mu &= \bar{\Lambda}_\rho{}^\sigma \frac{\partial}{\partial x^\sigma} x'^\mu \\
\delta^\mu_\rho &= \bar{\Lambda}_\rho{}^\sigma \frac{\partial}{\partial x^\sigma} \left(\Lambda^\mu{}_\nu x^\nu\right) \\
&= \bar{\Lambda}_\rho{}^\sigma \, \Lambda^\mu{}_\nu \frac{\partial x^\nu}{\partial x^\sigma} \\
&= \bar{\Lambda}_\rho{}^\sigma \, \Lambda^\mu{}_\nu \delta^\nu_\sigma \\
\delta^\mu_\rho &= \bar{\Lambda}_\rho{}^\sigma \, \Lambda^\mu{}_\sigma
\end{aligned}$$

In Matrix notation,:

$$\Lambda \, \bar{\Lambda}^T = \bar{\Lambda}\Lambda^T = \mathcal{I} \implies \bar{\Lambda} = \Lambda^{-T}.$$

However we just learned that Lorentz transformation matrices, i.e. the Λ's, in general are not orthogonal matrices, e.g. one can easily check this for the boost matrix $\Lambda(\beta \mathbf{x})$. Note that the equation,

$$\bar{\Lambda} = \Lambda^{-T} \tag{8}$$

when written in index notation gives us,

$$\bar{\Lambda}_\mu{}^\nu = \left(\Lambda^{-1}\right)^\nu{}_\mu \tag{9}$$

We will later observe that the bar in the notation for the inverse transformation, $\bar{\Lambda}$ will be unnecessary and we will get rid of it shorty.

- Define **contravariant vector** or **$(1, 0)$ rank tensor** as something which whose components would transform exactly in the manner the coordinate differences (differentials) do, i.e.

  $$V'^\mu = \Lambda^\mu{}_\nu \, V^\nu. \tag{10}$$

  Suppressing the indices we have in a component free matrix notation

  $$\begin{aligned}
  dx' &= \Lambda \, dx, \\
  V' &= \Lambda \, V.
  \end{aligned} \tag{11}$$

- Define **Covariant vector** or **$(0, 1)$ rank tensor** undergo the following change under a Lorentz transformation in a manner alike to the coordinate partials

  $$W_\mu \to W'_\mu = \bar{\Lambda}_\mu{}^\nu \, W_\nu. \tag{12}$$

- One can define higher rank tensors, e.g. a tensor of rank **(p,q)** which has p-upper indices and q-lower indices. e.g. $T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \ldots \nu_2}$. Such mixed rank tensor will transform with p-factors of the Lorentz transformation, Λ and q-factors of the inverse transformation, $\bar{\Lambda}$.

$$T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \ldots \nu_q} \to T'^{\mu_1 \cdots \mu_p}{}_{\nu_1 \ldots \nu_q} = \left(\underbrace{\Lambda^{\mu_1}{}_{\rho_1} \ldots \Lambda^{\mu_p}{}_{\rho_p}}_{p\text{-factors}} \right) \left(\underbrace{\bar{\Lambda}_{\nu_1}{}^{\sigma_1} \ldots \bar{\Lambda}_{\nu_q}{}^{\sigma_q}}_{q\text{-factors}} \right) T'^{\rho_1 \cdots \rho_p}{}_{\sigma_1 \ldots \sigma_q}$$

One such higher rank tensor which will prominent feature in our course is the Maxwell field strength tensor, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, which is a $\mathbf{(0, 2)}$ rank tensor.

- **Index contraction in products of vectors and tensors:** Consider the following combination which has no free indices,

$$W_\mu V^\mu.$$

How does this index-less object transform under a Lorentz transformation? Let's check,

$$\begin{aligned} W_\mu V^\mu \to W'_\mu V'^\mu &= \left(\bar{\Lambda}_\mu{}^\nu \, W_\nu\right) \left(\Lambda^\mu{}_\lambda \, V^\lambda\right) \\ &= \left(\bar{\Lambda}_\mu{}^\nu \, \Lambda^\mu{}_\lambda\right) \, W_\nu V^\lambda \\ &= \left(\underbrace{\left(\Lambda^{-1}\right)^\nu{}_\mu \, \Lambda^\mu{}_\lambda}_{=\delta^\nu_\lambda} \right) \, W_\nu V^\lambda \\ &= W_\nu V^\nu. \end{aligned}$$

Whenever we have such an expression with dummy indices, i.e. the same index appearing once upstairs and once downstairs, their Lorentz transformation cancel each other, we use the phrase that particular dummy index is **contracted**. This will be a generic feature, in general, when we will have a product of a certain number of tensors and/or vectors with some contracted (i.e. all repeated indices). The product will transform under a Lorentz transformation determined only by the free indices. E.g., the product,

$$A_{\mu\nu} B^\nu C^\mu{}_{\lambda\kappa}$$

will transform as if it only had the free indices λ and κ. The contracted indices μ and ν will not matter at all! (HW: Check this)

- **The Metric as an INVARIANT (0,2) rank tensor**: Let's write this invariant squared spacetime interval in 4-vector component notation,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx'^\mu dx'^\nu, \tag{13}$$

where we have introduced, the **Minkowski (flat) metric** $\eta_{\mu\nu}$. Comparing the RHS of (6) and

(13), we see that we have to set

$$
\begin{aligned}
\eta_{00} &= -1, \\
\eta_{11} = \eta_{22} = \eta_{33} &= +1, \\
\eta_{0i} = \eta_{i0} &= 0,\ i = 1,2,3 \\
\eta_{ij} &= 0,\ i \neq j.
\end{aligned}
$$

Right now there is no rigorous reason why we have chosen downstairs indices for η, because we have not checked whether it transforms like a $\mathbf{(0,2)}$-rank tensor yet! In fact, on the contrary we are attributing its components a fixed set of values, 0's and ± 1's, in *all inertial frames!* It seems like the metric does not transform at all. Right now we have written two downstairs indices for η is because the interval ds^2 is a scalar, and for a scalar all indices contracted would look nice and dandy!

Now we show that the metric is indeed a tensor despite not changing form or values from frame to frame. Not all objects with indices can qualify to be a tensor so we need to check each time whether an object with indices attached to it is transforming like a tensor would do under Lorentz transformation. Let's look at the metric, $\eta_{\mu\nu}$. We know in all frames it is identical i.e. has the same diagonal matrix form. But it also has two indices downstairs, so can it be a (0,2) type tensor? To check this we momentarily assume that the metric is a$\mathbf{(0,2)}$-rank tensor and then look at what it transforms to under a LT. If it remains same, then we can indeed call the metric a (0,2) tensor. So let's see what the metric transforms into when after we apply a LT assuming it is a (0,2) tensor. Using (12),

$$
\eta_{\mu\nu} \to \eta'_{\mu\nu} = \bar{\Lambda}_\mu{}^\alpha\, \bar{\Lambda}_\nu{}^\beta\, \eta_{\alpha\beta} \tag{14}
$$

If we look a bit hard at the RHS we see that, we can rewrite it a bit using (9),

$$
\begin{aligned}
\text{RHS} &= \bar{\Lambda}_\mu{}^\alpha\, \bar{\Lambda}_\nu{}^\beta\, \eta_{\alpha\beta} \\
&= \left(\Lambda^{-1}\right)^\alpha{}_\mu \left(\Lambda^{-1}\right)^\beta{}_\nu\, \eta_{\alpha\beta} \\
&= \left(\Lambda^{-T}\right)_\mu{}^\alpha\, \eta_{\alpha\beta}\, \left(\Lambda^{-1}\right)^\beta{}_\nu \\
&= \left(\Lambda^{-T}\eta\, \Lambda^{-1}\right)_{\mu\nu} \\
&= \eta_{\mu\nu}
\end{aligned}
$$

where in going from the second last line to the last line we had replaced $\eta \to \Lambda^T \eta\, \Lambda$ using the defining equation of Lorentz matrices (7). Thus, we have

$$
\eta_{\mu\nu} \to \eta'_{\mu\nu} = \eta_{\mu\nu}!
$$

So indeed the metric is an invariant $\mathbf{(0,2)}$-rank symmetric tensor.

- **Invariant scalar product** of two contravariant vectors: One can define a scalar i.e. Lorentz invariant quantity which is bilinear in two contravariant vectors V and W as follows

$$
V.W \equiv \eta_{\mu\nu} V^\mu W^\nu. \tag{15}
$$

Since all the tensor indices are contracted it is obvious that this product is a Lorentz invariant (scalar), but one can readily check the invariance i.e. show that $V.W = V'.W'$ using (10) and (14).

- **"Lowering the index": Covariant vectors revisited**: Looking at (15) sort of prompts us to invent a covariant object (quantity with a downstairs index) from a contravariant vector by contracting it with the metric tensor,
$$W_\mu = \eta_{\mu\nu} W^\nu.$$
Again it is obvious from the index structure of the RHS of above equation that the LHS is a $\mathbf{(0,1)}$-type or covariant vector. This general rule i.e. contraction by $\eta_{\mu\nu}$ allows us to convert contravariant/upstairs indices into covariant/downstairs ones. E.g.
$$A_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}A^{\alpha\beta}$$

- **Inverse of the metric**: We had already noted before that the Minkowski metric is self-inverse, $\eta^{-1} = \eta$. Does this mean the that we can express η^{-1} by the same component notation as η i.e. $\eta_{\mu\nu}$. The answer is negative! Lets look at the index structure on both sides of the following equation.
$$\eta^{-1}\eta = \mathcal{I}.$$
The RHS is identity matrix and has component notation, Kronecker delta: δ^μ_ρ [6]. Therefor the LHS must have an unsummed/free upstairs index μ and a downstairs free index ρ. If ν be the repeated index needed for matrix multiplication, then we have,
$$\left(\eta^{-1}\eta\right)^\mu_{\ \rho} = \left(\eta^{-1}\right)^{\mu\nu}\eta_{\nu\rho}$$
So the inverse must have BOTH indices upstairs. Since η is self-inverse we will drop the $(\ )^{-1}$, and **denote the inverse by$\boldsymbol{\eta^{\mu\nu}}$**.

- **"Raising the index" with inverse metric:** Now that we have the metric inverse with purely upstairs indices, we can use that to convert covariant stuff to contravariat stuff.
$$V^\mu = \eta^{\mu\nu}V_\nu.$$
Another example would be to convert the Maxwell tensor with covariant indices, $F_{\mu\nu}$ to contravariant indices, $F^{\mu\nu}$,
$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}.$$

- Finally we are ready to see why the "bar" in the covariant transformation matrix, $\bar{\Lambda}_\mu{}^\nu$ in (12) can be omitted without any fuss. First, multiplying both sides of (7) from the left by η^{-1}, we get,
$$\left(\eta^{-1}\Lambda^T\eta\right)\Lambda = \mathcal{I}$$

[6] This follows from the identity transformation: If $x' = x$, then the transformation matrix should be identity. In component notation,
$$\frac{\partial x'^\mu}{\partial x^\nu} = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu.$$

which means

$$\Lambda^{-1} = \eta^{-1}\Lambda^T\eta \tag{16}$$

Writing this is component form,

$$\left(\Lambda^{-1}\right)^{\mu}{}_{\nu} = \eta^{\mu\alpha}\,\Lambda^{\beta}{}_{\alpha}\,\eta_{\beta\nu}.$$

But since we have been using η and η^{-1} to lower and raise indices, we can do so for the RHS of the above equation and thus arrive at the equation,

$$\left(\Lambda^{-1}\right)^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}. \tag{17}$$

Comparing this with the definition of the $\bar{\Lambda}$-matrices in (9) we arrive at the conclusion,

$$\bar{\Lambda}_{\nu}{}^{\mu} = \Lambda_{\nu}{}^{\mu}!$$

Henceforth we will get rid of the bar and use the symbol, $\Lambda_{\mu}{}^{\nu}$ for covariant transformation matrices.

3 Homework Problems

- **Homework**: Lorentz Transformation of spacetime volume element,

$$d^4V = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.$$

transform under Lorentz transformation. Hint: The Wedge product (which defines the volume element) is defined by,

$$dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \equiv \varepsilon_{\mu\nu\rho\sigma} dx^{\mu} \otimes dx^{\nu} \otimes dx^{\rho} \otimes dx^{\sigma},$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita symbol in 4d (spacetime), with the convention $\varepsilon_{0123} = +1$. Use the above formula and the well known formula for the definition of a determinant of a matrix, $M_{n\times n}$ in terms of Levi-Civita symbols

$$\varepsilon_{\mu_1\ldots\mu_n} M^{\mu_1}{}_{\nu_1} M^{\mu_2}{}_{\nu_2} \ldots M^{\mu_n}{}_{n} = \det M\, \varepsilon_{\nu_1\ldots\nu_n}$$

to show that the volume element remains invariant.

- **Homework**: How does the D'Alembertian operator ("Box" operator) transform under Lorentz transformation,

$$\Box \equiv \partial^{\mu}\partial_{\mu} \equiv \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}.$$

- **Homework**: Show that the Kronecker delta, δ^{μ}_{ν} is an invariant tensor or type (1,1) and the Levi-Civita $\varepsilon_{\mu\nu\rho\sigma}$ is an invariant tensor of type (0,4).

4 Topology of the Lorentz Group

4.1 Orthochronous and Non-orthochronous Lorentz transformations

The defining equation for the Lorentz transformation matrices is,

$$\Lambda^T \eta\, \Lambda = \eta,$$

or in component notation,

$$\Lambda^\rho{}_\mu\, \Lambda^\sigma{}_\nu\, \eta_{\rho\sigma} = \eta_{\mu\nu}. \tag{18}$$

Since μ, ν are free indices, they can take any values. In particular consider the case when $\mu = \nu = 0$, i.e.,

$$\Lambda^\rho{}_0\, \Lambda^\sigma{}_0\, \eta_{\rho\sigma} = \eta_{00}.$$

Using $\eta_{00} = +1$, $\eta_{ij} = -\delta_{ij}$, we can simplify both sides and get,

$$\left(\Lambda^0{}_0\right)^2 - \left(\Lambda^i{}_0\right)^2 = 1 \Rightarrow \Lambda^0{}_0 = \pm\sqrt{1 + (\Lambda^i{}_0)^2}.$$

Thus, either $\Lambda^0{}_0 \geq +1$ or $\Lambda^0{}_0 \leq -1$. Since these two range of values for $\Lambda^0{}_0$ are non-overlapping, one cannot start from $\Lambda^0{}_0 \geq +1$ and smoothly reach a transformation, $\Lambda^0{}_0 \leq -1$ by continously changing some real parameter. In some sense these two ranges give us two diffrent/disjoint/unrelated Lorentz transformations. The Lorentz transformations for which $\Lambda^0{}_0 \geq +1$ are called ***orthochronous*** transformations, while those for which $\Lambda^0{}_0 \leq -1$ is called ***non-orthochronous***. Physically speaking, orthochronous transformations keep the direction of time same while non-orthochronous transformations reverse the direction of time. Examples of orthochronous Lorentz transformations are boosts, rotations, as well as parity and space inversion; while examples of non-orthochronous transformations are time-reversal or any product of a time-reversal operation with any number of boosts, rotations or parity/space inversion. Orthochronous Lorentz transformation matrices are denoted by $\Lambda^\uparrow$, while non-orthochronous Lorentz transformations are denoted by $\Lambda^\downarrow$.

4.2 Lorentz group has four disconnected components

We already noted before that Lorentz transformations could be either ***proper***, i.e. have unit determinant, $|\Lambda| = +1$, or be ***improper*** have determinant, $|\Lambda| = -1$. Since these two values of the determinants are also non-overlapping, it is not possible to start from one type, say a transformation of the kind $|\Lambda| = +1$, and reach a transformation of the other kind, $|\Lambda| = -1$ by continuously changing some real parameter. Thus topologically speaking these two subsets of the Lorentz group are disconnected in the space of all transformations. Following standard notation we shall refer to the set of proper Lorentz transformations by Λ_+ while the set of improper Lorentz transformations will be denoted by Λ_-. Thus in all Lorentz transformations can be divided into four following disconnected components (i.e. one cannot go from one component to the other by continuously changing some parameters),

- Proper Orthochronous $\left(\Lambda_+^\uparrow\right)$: e.g., boosts, rotations or products of any no. of boosts and rotations
- Improper Orthochronous $\left(\Lambda_-^\uparrow\right)$: e.g., Parity ($P$) or joint transformation of Parity and proper Orthochronous ($P\Lambda_+^\uparrow$)

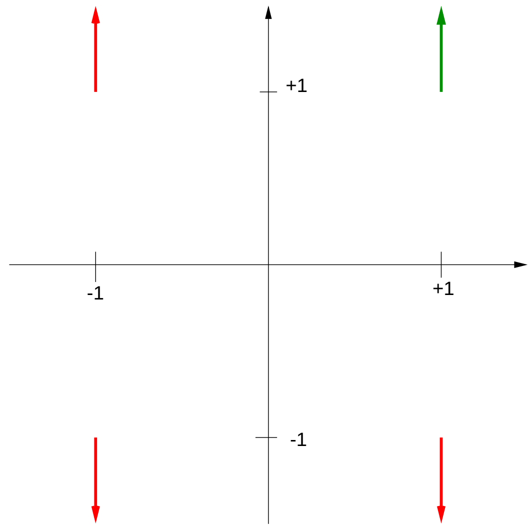

Figure 1: Lorentz group has four disconnected components in the space of matrices indicated by the red and green semi-infinite lines. The x-axis shows the determinant, $|\Lambda|$ while the vertical axis plots the element $\Lambda^0{}_0$. The component shaded in green is the restricted Lorentz group, $\Lambda_+^\uparrow$ or $SO^\uparrow(1,3)$. The components shaded in red are either improper, or Non-orthochronous or both.

- Improper Non-orthochronous $\left(\Lambda_-^\downarrow\right)$: e.g., Time-reversal ($T$) or joint transformation of time-reversal and a proper Orthochronous ($T\Lambda_+^\uparrow$)

- Proper Non-orthochronous $\left(\Lambda_+^\downarrow\right)$: e.g. PT or $PT\Lambda_+^\uparrow$.

These disconnected components of the Lorentz group are displayed in figure (1). The subset $\Lambda_+^\uparrow$ is continuously connected to the identity element and hence forms a group on its own right. This subgroup of the full Lorentz group is referred to as the ***Restricted Lorentz Group***. It is alternatively denoted by $SO^\uparrow(1,3)$ in some books.

5 Infinitesimal form of the (restricted) Lorentz transformations

Since the restricted Lorentz transformations are continuously connected to unity (i.e. they can be tuned to unity for some value of a continuous parameter, e.g, for the case of boosts by making the boost parameter/velocity zero), we can express them in the following *infinitesimal* form,

$$\Lambda^\mu{}_\nu \approx \delta^\mu_\nu + \omega^\mu{}_\nu. \tag{19}$$

Here $\omega^\mu{}_\nu$ are the infinitesimal or small boost velocity/ rotation angle. We have used an $\approx$ sign because we have dropped the $O(\omega^2)$ terms. Now using this infinitesimal form in the defining property of Lorentz transformation matrices (18) , we have

$$\left(\delta^\rho_\mu + \omega^\rho{}_\mu\right)\ \left(\delta^\sigma_\nu + \omega^\sigma{}_\nu\right)\ \eta_{\rho\sigma} = \eta_{\mu\nu}.$$

Expanding out the LHS,

$$\left(\delta_\mu^\rho + \omega^\rho{}_\mu\right)\ \left(\delta_\nu^\sigma + \omega^\sigma{}_\nu\right)\ \eta_{\rho\sigma} = \eta_{\mu\nu} + \omega_{\mu\nu} + \omega_{\nu\mu} + O(\omega^2).$$

Thus, equating linear order in ω on both sides we have,

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0. \tag{20}$$

So the matrix corresponding to covariant $\omega_{\mu\nu}$ is antisymmetric.

Further let's contract both sides of the anti-symmetry condition (20) by the inverse metric, namely, $\eta^{\mu\nu}$. This gives,

$$\eta^{\mu\nu}\omega_{\mu\nu} = 0,$$

or,

$$\omega^\nu{}_\nu = 0. \tag{21}$$

Thus, the mixed rank Lorentz parameter matrix, $\omega^\mu{}_\nu$ is traceless. This condition on ω can also be seen to arise from the condition of unit determinant,

$$\det\Lambda = +1.$$

For this we need the definition of the determinant,

$$\varepsilon_{\mu\nu\rho\sigma}\Lambda^\mu{}_0\Lambda^\nu{}_1\Lambda^\rho{}_2\Lambda^\sigma{}_3 = \det\Lambda, \tag{22}$$

and we plug in the infinitesimal form (19) to get to first order in ω,

$$1 + \varepsilon_{\mu\nu\rho\sigma}\omega^\mu{}_0\delta_1^\nu\delta_2^\rho\delta_3^\sigma + \varepsilon_{\mu\nu\rho\sigma}\delta_0^\mu\omega^\nu{}_1\delta_2^\rho\delta_3^\sigma + \varepsilon_{\mu\nu\rho\sigma}\delta_0^\mu\delta_1^\nu\omega^\rho{}_2\delta_3^\sigma + \varepsilon_{\mu\nu\rho\sigma}\delta_0^\mu\delta_1^\nu\delta_2^\rho\omega^\sigma{}_3 = 1$$

$$\implies \omega^0{}_0 + \omega^1{}_1 + \omega^2{}_2 + \omega^2{}_2 = 0. \tag{23}$$

i.e. $\omega^\mu{}_\nu$-matrix is traceless.

Homework: Find out $\omega^\mu{}_\nu$ and $\omega_{\mu\nu}$ corresponding to boosts and rotations. Also check (20) and (23) holds for them. If the general case, i.e. boost along a general direction and rotation along a general axis appear complicated, you may check for the simple case when the boost is along x-direction and rotation is around the z-axis.