Fall 2019: Classical Field Theory (PH6297)

Special Relativity reloaded*

August 13, 2019

1 Principle of covariance in special relativity and Poincaré invariance of the action

In class I mentioned that laws of physics (or equivalently equations of motion of physical systems) must have the "same form" in all inertial frames. From a practical point of view this means, one should be able to write down physical law/equations in the form,

$$T^{\cdots}_{\cdots} = 0.$$

The quantity on the left hand side, namely $T^{\cdots}_{\cdots}$, is some tensor (could be any generic(p, q)-rank tensor). This guarantees that if one performs a Lorentz transformation (or more generally a Poincaré transformation), the equation in the new inertial frame will be identical looking, i.e.,

$$T'^{\cdots}_{\cdots} = 0!$$

This is ensured because tensors transform homogeneously under a Lorentz transformation,

$$T'^{\cdots}_{\cdots} = \Lambda \ldots \bar{\Lambda} \ldots T^{\cdots}_{\cdots},$$

where Λ and $\bar{\Lambda}$ are the direct and inverse Lorentz transformation matrix elements.

So the next logical question is how do we go about writing down physical equations/laws in tensor form to begin with. To answer this question we recall that in general we obtain equations of motion for any physical system are obtained from an action (functional), I by applying the variational principle,

$$\delta I = 0.$$

The principle of covariance is then tantamount to demanding that the action, I itself is invariant under Poincaré transformation. One might ask, why does the action has to be a scalar (i.e. invariant) and not a tensor of a more general kind. The answer I believe cannot be found in the classical realm but instead lies in the quantum real where the action determines probability amplitudes of various processes, e^{iI}. If the action was a tensor (instead of a scalar), then such probability amplitudes will change when going from one inertial frame to another inertial frame. Thus, quantum mechanics demands us the action to be a

*Loosely based on material covered in lectures 6 and 7 (August 9 and 13) scalar under Lorentz (in fact Poincaré) transformations. **Thus, given a physical system our starting point should be an action which is a Lorentz scalar.** In addition we will demand the action to also satisfy some extra physical requirements. We list all of them them here

1. The action must be a scalar under Lorentz transformations and translations. (This is best understood when one realizes that in quantum theory the action for a given trajectory represents the probability amplitude, e^{iI} for that trajectory. Different inertial frames must compute the same probabilities for the same trajectory, hence one must have, $e^{iI} = e^{iI'}$. This will automatically hold if $I = I'$).

2. The action must be real. This has to do with probability conservation (unitarity) in quantum mechanics, complex values of action let to loss of unitarity. The amplitude e^{iI} will have a norm different than unity if $I \in \mathbb{C}$.

3. The action must involve at best two derivatives. Higher derivatives will lead to violation of causality or create instability (Ostrogradsky's theorem)

4. In case of field theories, the Lagrangian density should be local, $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$.

5. The action might be a scalar/invariant under other transformations (internal symmetries)

We will see that these conditions will be sufficient to construct action functional for physical theories, especially field theories of fundamental interactions in nature.

2 The Point Particle

The first physical system we will consider is not a field theory but the simplest possible system, namely the free (relativistic) point particle. In spacetime, the point particle traces out a one-dimensional curve which is known as the *worldline* of the point particle. Since a one-dimensional curve is usually defined by a single parameter, the spacetime coordinates of points on the worldline are functions of a single parameter[1]. So we introduce a "worldline parameter", λ which labels the worldline of the point particle i.e. all points on the world line are now functions of λ,

$$x^\mu = x^\mu(\lambda).$$

It is clear that one obvious Lorentz invariant quantity in this case is the length of the worldline, $\int ds$. Here ds^2 is the Lorentz invariant squared interval:

$$ds^2 = dx^\mu dx_\mu = \eta_{\mu\nu} dx^\mu dx^\nu.$$

Hence one can propose that the action of a point particle is,

$$I_{pp}[x^\mu(\tau)] \propto \int ds.$$

[1]Analogously a surface or sheet is two-dimensional the object, meaning the points on the surface are functions of two independent parameters, $x^\mu(\lambda_1, \lambda_2)$.

Since action has dimensions of angular momentum, one can fix the dimensionfull proportionality constant by introducing Lorentz invariant quantities such as the mass, m, and the speed of light, c

$$I_{pp}[x^\mu(\tau)] = \alpha\, mc \int ds,$$

where α is some dimensionless number. One can check that the action expression now has the correct dimensions. As you will find out in a homework exercise, $\alpha = -1$ for consistency with non-relativistic physics. Thus, we have the action for a free relativistic point particle,

$$I_{pp}[x^\mu(\tau)] = -mc \int ds = -mc \int \sqrt{dx^\mu dx_\mu}. \tag{1}$$

Since all three quantities, the mass, m, the signal speed, c and the ds, are Lorentz invariant, the action being the product of three Lorentz invariant quantities is itself a Lorentz invariant. Since the quantity within the square root is positive for $m^2 > 0$ particles, the action is real. The equations of motion are obtained by the principle of stationary action or least action - vary the action and set the variation to zero, $\delta I_{pp} = 0$. Since the point particle action is proportional to the four dimensional (spacetime) "path length", $\sqrt{ds^2}$, the principle of least (or stationary) action for the point particle is actually the "principle of shortest(stationary) path or distance".

Then we can rewrite the squared interval in parametric form as well:

$$ds^2 = dx^\mu(\lambda)\, dx_\mu(\lambda) = d\lambda^2\, \frac{dx^\mu(\lambda)}{d\lambda}\, \frac{dx_\mu(\lambda)}{d\lambda}. \tag{2}$$

So now the action looks like

$$I_{pp}[x^\mu(\tau)] = -mc \int d\lambda\, \sqrt{\dot{x}^\mu(\lambda)\dot{x_\mu}(\lambda)}. \tag{3}$$

where the overdot means total derivative wrt the worldline parameter, $\lambda : \dot{x}(\lambda) \equiv \frac{dx(\lambda)}{d\lambda}$.

As we shall shortly learn that reparameterization symmetry of the point particle action gives us the freedom to choose anything as a worldline parameter, let's use that to set the worldline parameter to be same as the lab frame time, t[2] i.e.

$$x^0(t) = ct, \mathbf{x}(t).$$

The parametric derivatives are then (lab frame) time derivatives, $\frac{dx^\mu}{dt}$. Now $\frac{dx^0}{dt} = c$ while the time-derivative of the spatial coordinates are,

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(t),$$

[2]Later we will opt for a different and more convenient form of parameterization called the proper time parametrization in which we choose the time in the frame attached to the point particle (say τ) itself as a parameter,

$$\lambda = \tau.$$

Since in the frame attached to the particle the particle does not move at all, $d\mathbf{x}' = 0$. So the invariant squared interval is,

$$ds^2 = c^2 d\tau^2 - d\mathbf{x}'^2 = c^2 d\tau^2.$$

i.e. the lab-frame velocity. Then, the parametric form of the squared interval (2) looks like,

$$ds^2 = dt^2 \frac{dx^\mu(t)}{dt}\frac{dx_\mu(t)}{dt}.$$

The μ-index is contracted i.e. summed over all $\mu = 0, 1, 2, 3$ and when we write out the sum explicitly, we get,

$$ds^2 = dt^2\left(c^2 - \mathbf{v}^2\right)$$

Then the action, (3) when written in terms of $t, \mathbf{x}$ in the lab-frame time parameterization, $\lambda = t$, becomes

$$I_{pp} = -mc^2 \int dt \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}, \text{ here } \quad \mathbf{v} = \frac{d\mathbf{x}}{dt}.$$

Homework: Show the above action then, becomes in the non-relativistic limit i.e. $c \to \infty$, or $\frac{\mathbf{v}^2}{c^2} \ll 1$, by expanding the square root binomially

$$I_{pp} = \int dt \left(\frac{1}{2}m\mathbf{v}^2\right) + (\text{x,v independent constant term}).$$

2.1 Equation of motion for the free relativistic point particle

Let's vary the action to get the equations of motion of the point particle. We can do this in two different ways. The first one I will use the Euler-Lagrange equations applied to the point particle Lagrangian thinking of the world line parameter, λ as time. Then,

$$L\left(x^\mu, \dot{x}^\mu, \lambda\right) = -mc\sqrt{\dot{x}^\mu \dot{x}_\mu}$$

where the $\{x^\mu\}$ are the generalized coordinates and the $\left\{\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}\right\}$ are the generalized velocities. The equation of motion for the coordinate x^μ is

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = \frac{\partial L}{\partial x^\mu}.$$

Plugging in $\frac{\partial L}{\partial x^\mu} = 0$, and the canonical momenta, $P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -\frac{mc\,\dot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}}$, we get the equation of motion to be,

$$\frac{d}{d\lambda}\left(\frac{c\dot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}}\right) = 0. \tag{4}$$

We can make things a little bit nicer in appearance by introducing a relativistic definition of velocity, "4-velocity" defined by

$$u_\mu \equiv \frac{c\dot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}}. \tag{5}$$

In terms of this the equation of motion of the point particle is

$$\frac{du_\mu}{d\lambda} = 0. \tag{6}$$

This is the equation of motion for the point particle for *arbitrary* paremeterization. So to get some insight we can ask how does it look in the lab frame where we can take the lab frame time, t to be the worldline parameter, i.e. $\lambda = t$. In that case the proper time velocity are,

$$u^\mu = \frac{c\dot{x}^\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}} = \left(\frac{c^2}{\sqrt{c^2 - \mathbf{v}^2}}, \frac{\mathbf{v}c}{\sqrt{c^2 - \mathbf{v}^2}} \right) = (\gamma c, \gamma \mathbf{v}) \, .$$

Hence we have,

$$\frac{d}{dt}\left(\frac{1}{\sqrt{1 - \frac{\mathbf{v^2}}{c^2}}} \right) = \frac{d}{dt}\left(\frac{\mathbf{v}}{\sqrt{1 - \frac{\mathbf{v^2}}{c^2}}} \right) = 0.$$

This immediately gives,

$$\frac{d\mathbf{v}}{dt} = 0,$$

as predicted by Newton's first law.

2.2 Proper time paremeterization : Straight lines trajectories

We can make things even more nicer in appearance if we choose a new parameter to relabel the world line. Say, we choose a new parameter. $\tau(\lambda)$ defined by integrating the following equation

$$\frac{d\tau(\lambda)}{d\lambda} = \sqrt{\dot{x}^\mu \dot{x}_\mu}$$

or,

$$d\tau = d\lambda \sqrt{\dot{x}^\mu \dot{x}_\mu}.$$

So in terms of τ we get the equation of motion to be,

$$\frac{d^2 x^\mu}{d\tau^2} = 0. \tag{7}$$

This new parameter, τ is in fact special and is equal to the proper time of the point particle (time in the rest frame of the particle), because the square of it is

$$d\tau^2 = (d\lambda)^2 \left(\frac{dx^\mu}{d\lambda} \frac{dx_\mu}{d\lambda} \right) = dx^\mu dx_\mu!$$

i.e. the squared interval. It is called proper time because, it has the same sign as t on the rhs,

$$d\tau^2 = dx^\mu dx_\mu = c^2 dt^2 - d\mathbf{x} \cdot d\mathbf{x}.$$

Solutions to this equation is of course straight lines,

$$x^\mu(\tau) = v^\mu \, \tau + w^\mu.$$

where, v, w are constants of integration.

2.3 Symmetries of the point particle action

A great virtue of the action formulation is that it reflects the symmetries of the system manifestly. For the point particle, we have

- Since the mass, m, τ are invariants, and all indices are summed over inside the square root, the action is manifestly Lorentz invariant i.e. under $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$.
- The action is in addition Poincare invariant. This is because the action only contains derivative of x, the derivative kills any constant shift, a

$$\frac{d(x+a)}{d\lambda} = \frac{dx}{d\lambda}$$

- The action is "Reparemeterization invariant" i.e. changing the parameter which is labeling the world line, $\lambda \to \lambda' = f(\lambda)$. This symmetry is distinguished from the above two by the fact that the symmetry transformation, $f(\lambda)$ is a function of of the worldline parameter, λ i.e. it varies along the length of the worldline. This is an example of a *gauge* symmetry[3].

2.4 Conserved Charges from the Action

Another virtue of the action is that one can easily deduce the conserved charges such as linear and angular momentum, energy, etc. very easily. This process of extracting conserved charges from the symmetries of the action is based on **Noether's theorem,** which is a very deep and fundamental result in physics, applicable in both classical and quantum system and which states:

Noether's Theorem: Corresponding to every continuous global symmetry of a physical system, there exists a conserved charge. If the symmetry is local (gauge), then there is no associated conserved charge. (Gauge symmetries represent redundancies of description)

From Sec. (2.3), we state the three conserved charges of the point particle:

Symmetries of the action	**Conserved Charges**
Lorentz symmetry (boosts, rotations), $\Lambda^\mu{}_\nu$	$M^{\mu\nu} = m\ (\dot{x}^\mu x^\nu - \dot{x}^\nu x^\mu)\,/\sqrt{\dot{x}^2}$
Translation symmetry, a^μ	$P^\mu = m\, c\dot{x}^\mu/\sqrt{\dot{x}^2} = m u^\mu$
Reparameterization symmetry, $\tau \to f(\tau)$	$\times$

Local continuous symmetries (gauge symmetries) represent redundancies in our description i.e. not all the generalized coordinates or velocities (momenta) used in the Lagrangian are independent and there are constraints relating them. E.g. for the point particle case, the reparameterization symmetry reflects the *mass-shell constraint*,

$$P_\mu P^\mu = m^2 c^2,$$

or equivalently,

$$u_\mu u^\mu = c^2$$

[3]Earlier $\Lambda^\mu{}_\nu$ and the shift a we constants, i.e. same all along the worldline. Such symmetries are called *global* symmetries.

Owing to this constraint condition, we clearly see that not all four components of canonical momenta (P^μ) or velocities (u_μ) are independent and determining three automatically fixes the fourth one.

Homework: Charge corresponding to global reparametrizations

Compute the *integral of motion* aka the conserved charge corresponding to the global shift symmetry of the worldline parameter,

$$\lambda \to \lambda' = \lambda + b$$

where b is a constant. Hint: Recall that in classical mechanics, for a system whose Lagrangian $L(q, \dot{q}, t)$ has no explicit dependence on time t, i.e. $L(q, \dot{q}, t) = L(q, \dot{q})$, there exists an conserved quantity (integral of motion) called the Hamiltonian given by,

$$H = \frac{\partial L}{\partial \dot{q}} \dot{q} - L.$$

In the point particle case, the worldline parameter λ plays the same role as t in usual classical mechanics, so the charge would be the worldline Hamiltonian.