Fall 2019: Classical Field Theory (PH6297)

Relativistic (tensor) fields, covariant fluids and covariant electrodynamics

August 15, 2019

1 Tensor fields

So far we have looked at a single 4-variable, x^μ, namely the position of single particle. This can be generalized to multiple but still finite particles with an index labeling the particle, $x_i^\mu(t)$. However consider the case when we keep on adding new degrees of freedom so that at the end we have an infinite number of variables, one at each point in space. Recall that such a collection of infinite number of variables at each point in space is called a **field**. For example, consider any point on a string strung along the horizontal direction, say along the x-axis. When the string is oscillating (transverse), at any instant of time, t each point on the string will have a vertical displacement in the y-direction. We can denote the vertical displacement by a field where we specify the location/point on the string, x and the time when it occurs, say t by a **displacement field**

$$y(x,t).$$

Similarly for longitudinal waves through a gas, at each point in the gas, we have a density fluctatuon field,

$$\Delta\rho(\mathbf{x},t)$$

which is the change in density of the material from its normal equillibrium density.

Generalizing, we can define a “tensor fields” which means at each point in space we have a tensor-valued quantity which is dependent on time. For example, a scalar field, $\phi(x^\mu)$ or a vector field, $V^\mu(x)$. Then it is natural to ask what is the transformation law for such tensor fields under a Lorentz transformation,

$$x \to x' = \Lambda x.$$

The answer becomes obvious after a little thought,

$$\phi(x) \to \phi'(x') = \phi(x),$$

$$V^\mu(x) \to V'^\mu(x') = \Lambda^\mu{}_\nu\, V^\nu(x),$$

$$\omega_\mu(x) \to \omega'_\mu(x) = \Lambda_\mu{}^\nu \omega_\nu(x).$$

These transformations are often called the **passive transformations.**

Often, we shall be interested in the **active transformation** of a tensor field after an LT, defined by,

$$\begin{aligned}
\Delta V^\mu(x) &= V'^\mu(x) - V^\mu(x). \\
&= \Lambda^\mu{}_\nu\, V^\nu(\Lambda^{-1}x) - V^\mu(x) \\
&= \Lambda^\mu{}_\nu\, V^\nu\left(\Lambda_\beta{}^\alpha\, x^\beta\right) - V^\mu(x) \\
&= (\delta^\mu_\nu + \omega^\mu{}_\nu)\; V^\nu\left(x^\alpha + \omega_\beta{}^\alpha\, x^\beta\right) - V^\mu(x^\alpha) \\
&= \omega^\mu{}_\nu\, V^\nu(x) + \omega_\beta{}^\alpha\, x^\beta\, \partial_\alpha V^\mu(x^\alpha) \\
&= \omega^\mu{}_\nu\, V^\nu(x) + \omega^{\alpha\beta}\, \mathcal{M}_{\alpha\beta} V^\mu, \qquad \mathcal{M}_{\alpha\beta} = [x_\alpha\, \partial_\beta - x_\beta\, \partial_\alpha]
\end{aligned}$$

Observe that $\mathcal{M}$ is similar form as the angular momentum operator in quantum mechanics (except the i's are missing).

2 Covariant Fluids

The fluid field is parametrized by the density, $\rho(\mathbf{x}, t)$ and the current $\boldsymbol{j}(\mathbf{x}, t) = \rho(\mathbf{x}, t)\, \boldsymbol{v}(\mathbf{x}, t)$, where $\boldsymbol{v}(\mathbf{x}, t)$ is the fluid velocity at $\mathbf{x}$ and at time t. Now, a well known equation for the fluid flow which represents conservation of mass is the continuity equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0.$$

In the context of a fluid we usually talk about two frames, again the Lab frame, S and the fluid frame, S' which is attached to a fluid element. Let's choose the direction of velcoty of the fluid element to be the common X-axis of both frames. Now let's look at the expression for the density. In the lab frame this is,

$$\rho = \frac{\Delta m}{dx\, dy\, dz}.$$

In the fluid frame, S', this is,

$$\rho' = \frac{\Delta m}{dx'\, dy'\, dz'}.$$

Since in the Lab frame the moving fluid element will appear to be Lorentz contracted along the x-direction,

$$dx = \frac{dx'}{\gamma},$$

while transverse length/dimensions of teh fluid element are unchanged,

$$dy = dy', \quad dz = dz'.$$

This imples one can write

$$\begin{aligned}
\rho &= \gamma \rho', \\
j_x &= \beta c \rho \\
&= \gamma\beta(c\rho').
\end{aligned}$$

Since the velocity of the fluid in the fluid rest frame is zero, we have, $j'_x = 0$. So we can write down the following pair of equations,

$$\rho = \gamma\rho' + \gamma\frac{\beta}{c}j'_x,$$
$$j_x = \gamma j'_x + \gamma\beta(c\rho').$$

In matrix form,

$$\begin{pmatrix} c\,\rho \\ j_x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} c\,\rho' \\ j'_x \end{pmatrix},$$
$$\begin{pmatrix} c\,\rho' \\ j'_x \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} c\,\rho' \\ j'_x \end{pmatrix}$$

This equation is identical to the Lorentz transformation for the position 4-vector. From this one can deduce the existence of a new four vector for the fluid, the "4-current",

$$j^\mu = \left(j^0, \boldsymbol{j}\right), j^0 = c\rho.$$

The equation of continuity can then be written in a frame invariant form,

$$\partial_\mu j^\mu = 0$$

with equal number of up and down indices. (We will revisit relativistic fluid dynamics in detail in Gravitation and Cosmology course)

2.1 Electric current density

For a single point charge moving in a trajectory, $\mathbf{x}(t)$, we have the charge density,

$$\rho\left(t, \mathbf{x}\right) = q\,\delta^3\left(\mathbf{x} - \mathbf{x}(t)\right), \tag{1}$$

while the current density is,

$$\mathbf{j}(t, \mathbf{x}) = \rho(t, \mathbf{x})\,\mathbf{v}(t) = q\,\mathbf{v}(t)\,\delta^3\left(\mathbf{x} - \mathbf{x}(t)\right) \tag{2}$$

Of course both these expressions look non-covariant and we need to recast it in a form where the Lorentz transformation is manifest i.e. as the components of a four vector, j^μ. One could simply replace every quantity in expressions (1, 2) by their respective four-vector counterpart, leading to the tentative expression,

$$j^\mu(x^\nu) \stackrel{?}{=} q\,u^\mu(\tau)\,\delta^4\left(x^\nu - x^\nu(\tau)\right), \tag{3}$$

where τ is some invariant parameter finally to be indentified with the proper time. However note that we have one extra Dirac delta compared to the non-relativistic expression and instead of lab frame time, t the parameter is τ. So this cannot be the correct answer. Let's try to guess the answer by switiching back to t as a worldline parameter, then by comparing the rhs for $\mu = i$ i.e. by comparing to the rhs of (2). We have the rhs of the tentative expression (3),

$$\begin{aligned}
\text{RHS} &= q\,\frac{dx^i(\tau)}{d\tau}\,\delta(t-t(\tau))\,\delta^3\left(\mathbf{x}-\mathbf{x}(\tau)\right) \\
&= q\,\frac{dx^i(\tau(t))}{dt}\,\frac{dt}{d\tau}\,\delta(t-t(\tau))\,\delta^3\left(\mathbf{x}-\mathbf{x}(\tau)\right) \\
&= q\,\frac{dx^i(\tau(t))}{dt}\,\delta(\tau-\tau(t)\,\delta^3(\mathbf{x}-\mathbf{x}(\tau(t))) \\
&= q\,\frac{dx^i(t)}{dt}\,\delta(\tau-\tau(t))\,\delta^3(\mathbf{x}-\mathbf{x}(t)).
\end{aligned}$$

So now we can identify what is missing and what to do in order for the RHS to be same as the RHS of (2). We just need to integrate over τ! So we have the expression for the current density four-vector for a point electric charge to be,

$$j^\mu(x) = q\int d\tau\; u^\mu(\tau)\,\delta^4\left(x^\nu - x^\nu(\tau)\right). \tag{4}$$

3 Classical Electrodynamics in Lorentz covariant form

The source of electric and magnetic fields is the charge density and current density. We already know how to fuse them into a single relativistic entity, the 4-current,

$$j^\mu = (c\rho, \boldsymbol{j}).$$

The rest is to figure out how do **E, B** combine into some relativistic format. The Maxwell's equation in terms of $\mathbf{E}, \mathbf{B}$

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \rho, \qquad \boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{1}{c}\boldsymbol{j}, \tag{5}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0. \tag{6}$$

The two equations in the first line (5)are similar in one respect, they both have the sources ρ and $\mathbf{j}$ on their RHS. Since the sources $\rho, \mathbf{j}$ can be combined into a single 4-vector, j^μ, namely the current density 4-veector, perhaps the first two Maxwell equations can be combined into a single 4-vector equation. However one thing is clear, there is a mismatch of number of degrees of freedom on the LHS (**E** and **B** have $3+3=6$ components) and RHS (source j^μ have 4 components). So we are missing something right now and a Lorentz covariant form is not obvious. To make further progress, we will need to recall our old friends, the scalar and vector potentials,

$$\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \\
\mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A}.
\end{aligned} \tag{7}$$

Now it is pretty obvious that one can combine the scalar and vector potential into a single 4-vector potential, usually called the **Maxwell Gauge field**,

$$A^\mu = (\Phi, \mathbf{A}).$$

This means the covariant 4-potential is, $A_\mu = (\Phi, -\mathbf{A})$.

Now looking at (7) seems to indicate now needs to act with the derivates on the gauge field to get E and B. Can we try something like $\mathbf{E}, \mathbf{B} \sim \partial_\mu A_\nu$? The answer is no because the number of components on both sides don't match up. $\mathbf{E}$ and $\mathbf{B}$ have a total of 6 components while $\partial_\mu A_\nu$ has $4 \times 4 = 16$ components (the indicesμ and ν can take 4 values each namely 0,1,2,4 , so the total number of index combination is 4×4). Clearly we have to select only a 6 component subset of these 16 derivatives but at the same time it has to be an "irrep" of Lorentz group. One possibility is to make a $(0, 2)$ type tensor, $F_{\mu\nu}$ defined by,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The $F_{\mu\nu}$ is called the (Maxwell) **field strength tensor**. This being manifestly antisymmetric in the two indices has ${}^4C_2 = 6$ components as well! Indeed one can check that,

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = -\frac{1}{c}\frac{\partial A^i}{\partial t} - \frac{\partial \Phi}{\partial x^i} = E^i.$$

Similarly, one can show,

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \partial_0 A^i + \partial_i A^0 = \frac{1}{c}\frac{\partial A^i}{\partial t} + \frac{\partial \Phi}{\partial x^i} = -E^i.$$

Homework: Show how F_{ij} is related to the magnetic field. Then write out $F_{\mu\nu}$ with the question marks replaced by stuff in terms of B.

$$\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & ? & ? \\ E_y & ? & 0 & ? \\ E_z & ? & ? & 0 \end{pmatrix}.$$

Now the first Maxwell equation with sources can be expressed in a covariant way in terms of the field strength, $F_{\mu\nu}$ and current density 4-vector, j_μ as

$$\partial^\mu F_{\mu\nu} = \frac{1}{c} j^\nu. \tag{8}$$

The second line of Maxwell equations (6) contains two equations which are homogeneous i.e. no sources of charges or currents on their right hand sides. How can we rewrite these equations in a Lorentz covariant notation. To find out how, let's first look at the homogeneous equation,

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0.$$

This can be written as,

$$\partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3 = 0.$$

But recall, $B^1 = F_{23}$, $B^2 = F_{31}$, and $B^3 = F_{12}$. Thus this equation, $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, becomes

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0.$$

With a little more thought it becomes clear that *both* the homogenous equations can be expressed as,

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{9}$$

This homogeneous equation is called the **Bianchi identity** for Maxwell theory. These equations are a bit weird because they do not care about what the source is - could be a point charge or a line charge or some strange distribution of currents. Thus, they are not equations of motion, but instead represent some constraints/ consistency conditions on the physically allowed $\mathbf{E}$ and $\mathbf{B}$ fields.

There is another equivalent way of writing down the homogenous Maxwell equations (6) in a Lorentz covariant fashion. For this notice that these combined together are four equations (the first one is a scalar equation the second one is a 3-vector equation). So this hints at the fact that these can be perhaps four components of a single 4-vector equation? But if we look at the LHS of these equations we see that we have spacetime derivatives of the $\mathbf{E}, \mathbf{B}$ fields i.e. things of the form $\partial_\mu F_{\nu\rho}$. A term like $\partial_\mu F_{\nu\rho}$ has 3 indices and to make a 4-vector it seems we will have to contract 2 of the 3 indices. But there is only one obvious way to do that, namely the $\partial^\mu F_{\mu\rho}$, and we have already used that up while covariantizing the first line of Maxwell equations (5). So we have to think hard to come up with a different way to contract indices. This can be done by introducing a **dual field strength** tensor,

$$G_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{10}$$

then the homogeneous Maxwell's equations (6) can be expressed in terms of the dual field strength tensor as,

$$\partial^\mu G_{\mu\nu} = 0. \tag{11}$$

3.1 Action for Electrodynamics

The action again must be made out Lorentz invariant quantities, like for example, $A_\mu A^\mu$, $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}G^{\mu\nu}$, $G_{\mu\nu}G^{\mu\nu}$, $j^\mu A_\mu$ etc. From this list we will need just two terms,

$$I[A] = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}j_\mu A^\mu\right). \tag{12}$$

The curious factor of $-\frac{1}{4}$ is so that when we expand out the $F_{\mu\nu}F^{\mu\nu}$ term, we get a positive kinetic term with the correct coefficient for the potential, to wit, $\frac{1}{2}\dot{\mathbf{A}}\cdot\dot{\mathbf{A}}$.

Homework: Check that this action gives you (8). Note that the Bianchi identity (9) cannot be extracted from the action because it is NOT an equation of motion.

Homework: Check the term $G_{\mu\nu}G^{\mu\nu}$ is nothing new because it is proportional to $F_{\mu\nu}F^{\mu\nu}$.

Homework: Check the term $G_{\mu\nu}F^{\mu\nu}$ is a total derivative i.e. of the form, $\partial_\mu(T^{\mu\cdots}{}_{\nu\ldots})$.

Note: The term $A_\mu A^\mu$ is not allowed because it breaks gauge invariance. Such a term would mean photon has a mass. We know that the photon has zero mass.

Homework: Gauge Invariance of the Maxwell field

The Electric field and magentic field are invariant under a special redefinition of the potentials Φ, $\mathbf{A}$, namely,

$$\begin{aligned}\Phi \to \Phi' &= \Phi + \partial_t \chi, \\ \mathbf{A} \to \mathbf{A}' &\to \mathbf{A} + \boldsymbol{\nabla}\chi,\end{aligned}$$

for an *arbitrary* parameter χ. This can be expressed in a covariant form,

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi. \tag{13}$$

This is called a gauge transformation[1]. Show that under such a transformation $F_{\mu\nu}$ remains unchanged i.e. it is gauge invariant. (This means that the Maxwell term, $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is gauge invariant as well). Show that the term, $j_\mu A^\mu$ is also invariant up to a total derivative for a *conserved* current j_μ i.e. when continuity $\partial_\mu j^\mu = 0$ holds. Show that a term like $A_\mu A^\mu$ is not invariant under gauge transformations (13).

3.2 Charged point particle in an Electromagnetic field

Let consider a point particle of mass, m and electric charge, q (a point electric monopole) in a background electro-magnetic field, $F_{\mu\nu}$. We know that in the non-relativistic/Newtonian limit, we should get an equation of motion,

$$m\,\frac{d\mathbf{v}}{dt} = q\,\mathbf{E} + q\,\left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right). \tag{14}$$

We also know from the (neutral) point particle case in Sec. **??**, the lhs of the above equation can be made covariant (i.e. a four vector/tensor) as follows,

$$m\,\frac{d\mathbf{v}}{dt} \to m\,\frac{du^\mu}{d\tau}.$$

The rhs is not that obvious since it has a two index object, $q\,F^{\mu\nu}$. So to make it match up with the lhs in terms of indices, we need to contract $F^{\mu\nu}$ with some (till now undetermined) vector, t_μ.If we look at the rhs of the noncovariant law, (14) we see the presence of a velocity and we are lead to suspect that this vector could be $u_\mu(\tau)$, the four-velocity vector. Indeed that is the correct guess. So now we can contract $F^{\mu\nu}$ on the rhs with $u_\nu(\tau)$ and we get the fully covariant version of the Lorentz force law,

$$m\,\frac{du^\nu(\tau)}{d\tau} = \frac{q}{c}\,F^{\mu\nu}(x(\tau))\,u_\nu(\tau). \tag{15}$$

What about the action, I_{pp}? How does it change from the *free* case? The answer can again be arrived at from a symmetry perspective. We can write down some terms which are Lorentz invariant *and* gauge invariant,

$$I_{EMp} = q\int d\tau\; F^{\mu\nu}\,u_\mu\,u_\nu.$$

[1] To be precise this is a $U(1)$ gauge transformation.

But this vanishes, being a product of antisymmetric and symmetric tensors, so we might try,

$$I_{EMp} = q \int d\tau \, u_\mu \, A^\mu .$$

Homework: Show that if you add a term

$$I_{EMp} = q \int d\tau \, u_\mu(\tau) \, A^\mu(x(\tau))$$

to the free point particle action, $I_{pp} = -mc \int \sqrt{dx^\mu dx_\mu}$, then we indeed get back the covariant Lorentz force law (15) upon varying the full action, $I_{pp} + I_{EMp}$.

4 Levi-Civita tensor

We introduced the Levi-Civita *symbol*[2], $\varepsilon_{\mu\nu\rho\sigma}$, which is completely antisymmetric in it's indices, i.e. swapping any two indices turns it negative wrt to what it was. We will work with the convention,

$$\varepsilon_{0123} = +1.$$

Since it is completely antisymmetric by definition, it follows that in general,

$$\begin{aligned} \varepsilon_{\mu\nu\rho\sigma} &= (-)^P \varepsilon_{0123}, \\ &= (-)^P. \end{aligned}$$

where P and is the number of exchanges/swaps one needs to get to $\mu\nu\rho\sigma$ from 0123[3].

Example: Consider the element ε_{2130}. We need to do the following two swaps to arrive at this element from ε_{0123}

$$\varepsilon_{\underline{0}12\underline{3}} \to \varepsilon_{3120},$$

followed by,

$$\varepsilon_{\underline{3}1\underline{2}0} \to \varepsilon_{2130}.$$

Since we needed to do two swaps, $P = 2$, and we have,

$$\varepsilon_{2130} = (-)^2 = +1.$$

Example: A cyclic change of all four indices, 0123 → 1230 → 2301 → 3012 requires **odd** number of swaps. Lets look at the1230 case. We arrive at this from 0123 using the following steps,

$$\begin{aligned} \underline{0}12\underline{3} &\to 3120, \\ \underline{3}1\underline{2}0 &\to 2130, \\ \underline{21}30 &\to 1230. \end{aligned}$$

[2]We are calling it a symbol since we have not proven yet that it is a tensor.

[3]Usually P is defined as the number of swaps modulo 2, i.e. it can be either 0 (even) or 1 (odd), but it makes no difference if we omit this modulo 2.

So it required 3 swaps i..e. $P = 3$, and hence,

$$\varepsilon_{1230} = (-)^3 = -1.$$

Example: A cyclic change of three out of four indices requires **even** number of swaps. Consider the case of $0123 \to 0231$. One needs the following two swaps to attain this

$$\begin{aligned} 0\underline{1}2\underline{3} &\to 0321 \\ 0\underline{32}1 &\to 0231. \end{aligned}$$

Since, $P = 2$, we have,

$$\varepsilon_{0231} = +\varepsilon_{0123}.$$

One can define a new symbol with all upstairs indices by raising all four indices of the Levi-Civita symbol, *viz.*

$$\varepsilon^{\mu\nu\rho\sigma} = \eta^{\mu\alpha}\eta^{\nu\beta}\eta^{\rho\gamma}\eta^{\sigma\delta}\varepsilon_{\alpha\beta\gamma\delta}.$$

This implies,

$$\begin{aligned} \varepsilon^{0123} &= \eta^{0\alpha}\eta^{1\beta}\eta^{2\gamma}\eta^{3\delta}\varepsilon_{\alpha\beta\gamma\delta} \\ &= (\det\eta) \\ &= -1. \end{aligned}$$

In general,

$$\underline{\varepsilon^{\mu\nu\rho\sigma}} = -\underline{\varepsilon_{\mu\nu\rho\sigma}},$$

where the underline means this equation is not a tensor equation (since indices are mismatched on both sides) but just an equation reflecting the equality of numerical value of components of the two tensors on the two sides.

4.1 Levi-Civita Symbol under Lorentz transformation: Levi-Civita tensor

Let's see what the Levi Civita symbol turns into after a Lorentz transformation,

$$\begin{aligned} \varepsilon'_{\mu\nu\rho\sigma} &= \Lambda_\mu{}^\alpha\Lambda_\nu{}^\beta\Lambda_\rho{}^\gamma\Lambda_\sigma{}^\sigma\ \varepsilon_{\alpha\beta\gamma\delta} \\ &= (\det\Lambda)\,\varepsilon_{\mu\nu\rho\sigma} \end{aligned}$$

For proper Lorentz transformations, $\det\Lambda = 1$, so Levi-Civita remains the unchanged and it is an invariant tensor of type $(0,4)$! However for improper Lorentz transformations, $\det\Lambda = -1$ and Levi-Civita does not transform as a $(0,4)$-rank tensor.

4.2 Identities on contractions of Levi-Civita tensor

4.2.1 $\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}$

All indices are dummy/contracted so one has to sum over all possible distinct μ, ν, ρ, σ. There are in all $4! = 24$ such terms and since $\underline{\varepsilon^{\mu\nu\rho\sigma}} = -\underline{\varepsilon_{\mu\nu\rho\sigma}}$, each term is equal to -1, so we have,

$$\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} = -4!. \tag{16}$$

4.2.2 $\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\rho\tau}$

Now since the Levi-Civita is totally antisymmetric, for the contracted product to be non-vanishing, σ and τ have to take the same value. For example, if $\mu = 0, \nu = 1, \rho = 2$, then σ and τ both have to be 3 as there is no other choice. So we can write down,

$$\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\rho\tau} \propto \delta^{\tau}_{\sigma}.$$

Now we have to find out the proportionality constant, call it C

$$\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\rho\tau} = C\ \delta^{\tau}_{\sigma}.$$

This is very easily determined by giving special values to the free indices σ, τ, say $\sigma = \tau = 3$, then we have,

$$C = \varepsilon_{\mu\nu\rho 3}\varepsilon^{\mu\nu\rho 3}.$$

Now given that the fourth index is 3, the first three indices μ, ν, ρ can only take values $0, 1, 2$. There are in total 3! such terms (permutation of $0, 1, 2$). Also recall that,

$$\underline{\varepsilon^{\mu\nu\rho 3}} = -\underline{\varepsilon_{\mu\nu\rho 3}}.$$

So we have,

$$C = -3!,$$

and the identity,

$$\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\rho\tau} = -3!\ \delta^{\tau}_{\sigma}. \tag{17}$$

4.2.3 $\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\alpha\beta}$

Again looking at the index structure this quantity is $(2, 2)$ type tensor and again as before we note that the antisymmetry of the Levi-Civita means that once we fix μ and μ then either $\rho = \alpha, \sigma = \beta$ or $\rho = \beta, \sigma = \alpha$, i.e.

$$\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\alpha\beta} = D\ \left(\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma} - \delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma}\right). \tag{18}$$

The negative sign gets chosen because α and β are antisymmetric indices. Now we need to determine the unknown constant, D and we proceed as we did in the previous case, i.e. by consider special values of the free indices,

$$\rho = \alpha = 2, \sigma = \beta = 3.$$

We get,

$$
\begin{aligned}
D &= \varepsilon_{\mu\nu 23}\varepsilon^{\mu\nu 23} \\
&= \varepsilon_{0123}\varepsilon^{0123} + \varepsilon_{1023}\varepsilon^{1023} \\
&= -2.
\end{aligned}
$$

So we have,

$$
\begin{aligned}
\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\alpha\beta} &= -2! \left(\delta^\alpha_\rho \delta^\beta_\sigma - \delta^\alpha_\rho \delta^\beta_\sigma\right). \\
&= -2!\; 2!\; \delta^{[\alpha}_\rho \delta^{\beta]}_\sigma.
\end{aligned}
$$

Here we have introduced the "antisymmetrized sum" defined by,

$$
\begin{aligned}
[AB] &= \frac{1}{2!}\left(AB - BA\right), \\
[ABC] &= \frac{1}{3!}\left(ABC + BCA + CAB - ACB - BAC - CBA\right).
\end{aligned}
$$

and so on.

Optional Homework: Simplify the following product of two Levi-Civita's with a single index contracted $\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\alpha\beta\gamma}$.

All of these identities with contractions of course can be obtained from the mother of all identities,

$$
\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta} = -4!\; \delta^{[\alpha}_\mu \delta^\beta_\mu \delta^\gamma_\rho \delta^{\delta]}_\sigma.
$$