Fall 2019: Classical Field Theory (PH6297)

Scalar Field Theory I*

August 21, 2019

1 An action functional and the equation of motion for scalar field

Here we consider the field theory where the degree of freedom is a real scalar field, $\varphi(x)$. The first step is to construct an action functional, $I[\varphi(x)]$. The action must be real, as well as a Lorentz scalar, and since this is a field theory the action must be a spacetime integral of a lagrangian density,

$$I[\varphi(x)] = \int d^4x\, \mathcal{L}(x),$$

where the lagrangian density is itself a Lorentz scalar and a **local** function of the field φ and the first derivative(s) $\partial_\mu \varphi$,

$$\mathcal{L}(x) = \mathcal{L}\left(\varphi(x), \partial_\mu \varphi(x)\right).$$

Since the lagrangian density is a Lorentz scalar, one can allow terms in the lagrangian density an arbitrary function of the scalar field $\varphi(x)$ itself, say $V(\varphi)$. Now terms in the lagrangian which will contain derivatives of the field, such as $\partial_\mu \varphi$, one needs to construct scalar by contracting the Lorentz indices. One such term is $(\partial_\mu \varphi)(\partial^\mu \varphi)$. However generically one can have more general terms such as $K(\varphi)(\partial_\mu \varphi)(\partial^\mu \varphi)$, where $K(\varphi)$ is an arbitrary function of the scalar field. So it seems the general lagrangian is of the form,

$$\mathcal{L} = K(\varphi)(\partial_\mu \varphi)(\partial^\mu \varphi) - V(\varphi).$$

The first term containing derivatives will be referred to as the *kinetic energy* term,

$$T = K(\varphi)(\partial_\mu \varphi)(\partial^\mu \varphi).$$

Now one can always redefine the scalar field (which is a canonical transformation in classical mechanics). For example, one can define a new field, ϕ,

$$\partial_\mu \phi \sim \sqrt{K(\varphi)}\partial_\mu \varphi.$$

This is a first order PDE and can be integrated under some general conditions. In terms of the new field the kinetic term becomes simple,

$$T \sim \partial_\mu \phi\, \partial^\mu \phi.$$

*Notes for Lecture 10 (Aug. 20, 2019)

For now on we will always consider a *canonically normalized* which has a factor of $\frac{1}{2}$,

$$T = \frac{1}{2}\partial_\mu\varphi \, \partial^\mu\varphi.$$

With this canonically normalized term, the lagrangian for a real scalar field theory is,

$$\mathcal{L} = T - V = \frac{1}{2}\partial_\mu\varphi \, \partial^\mu\varphi - V(\varphi).$$

Next we write down the Euler-Lagrange equation for this real scalar field,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \varphi\right)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi}$$

Inserting, $\frac{\partial \mathcal{L}}{\partial \varphi} = -\frac{\partial V}{\partial \varphi}$ and $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \partial^\mu \varphi$ respectively, one gets the equation of motion,

$$\Box\varphi = -\frac{\partial V}{\partial \varphi},$$

or,

$$\Box\varphi + \frac{\partial V}{\partial \varphi} = 0.$$

where we are denoting $\partial_\mu \partial^\mu \equiv \Box$.

Now to consider physically interesting theories we need to specialize or restrict ourselves to some specific forms of the potential function, $V(\varphi)$. One physical such restriction is that the potential should be an analytic function in φ. At the level of classical theory one might not be able to appreciate the significance of this restriction, but in quantum theory it will become evident that non-analytic potentials or terms in the action will imply or result in breakdown of effective field theory description at some regions of moduli space. Foreseeing this situation we will strictly consider analytic potentials, $V(\varphi)$ and since analytic functions as a power series,

$$V(\varphi) = \sum_{n=0}^{\infty} a_n \, \varphi^n = a_0 + a_1\varphi + a_2\,\varphi^2 + a_3\,\varphi^3 + a_4\,\varphi^4 + \ldots.$$

We can safely drop or ignore a constant piece a_0 since such a constant term in the lagrangian does not contribute to the equation of motion,

$$V(\varphi) = a_1\varphi + a_2\,\varphi^2 + a_3\,\varphi^3 + a_4\,\varphi^4 + \ldots.$$

Now the odd-power terms, $V_{odd} = a_1\varphi$, $a_3\varphi^3$ etc. have a pathological property, namely, $V_{odd} \to -\infty$ as $\varphi \to \infty$. Thus if we keep such odd power terms in the potential energy function, V, they will lead to the potential having no lower bound. Systems which have potentials with no lower bound have no stable ground state, and hence are considered nonphysical. Since we are interested with physical systems we will omit such odd powered terms in the potential function,

$$V(\varphi) = a_2\,\varphi^2 + a_4\,\varphi^4 + \ldots.$$

For simplicity we will restrict ourselves to quartic terms,

$$V(\varphi) = a_2\,\varphi^2 + a_4\,\varphi^4,$$

and rechristen the coefficients, $a_2 = \frac{m}{2}$, and $a_4 = \frac{\lambda}{4!}$, to make the potential look consistent with the literature,

$$V(\varphi) = \frac{m}{2}\,\varphi^2 + \frac{\lambda}{4!}\,\varphi^4.$$

The quadratic piece, $\frac{m}{2}\,\varphi^2$ is referred to as the "*mass term*" and the quartic piece, $\frac{\lambda}{4!}\,\varphi^4$ is referred to as the "*(self) coupling term*" or "*(self) interaction term*". The nomenclature will become obvious in the next few sections. It will turn out this restriction to quadratic and quartic potentials will cover a large class of physical systems especially in particle physics and statistical mechanics and is a widely studied system which goes by the name of "$\lambda\varphi^4$ **theory**". Terms with higher powers of φ, are also undesirable in a fundamental theory since they will lead to quantum theory with no predictive power (non-renormalizable), so we will not consider them. However, for effective field theories such nonrenormalizable terms are fine.

The equation of motion for the λ-φ^4 theory is,

$$\Box\varphi + m^2\varphi + \frac{\lambda}{3!}\varphi^3 = 0,$$

or as is more customarily presented in books,

$$\left(\Box + m^2\right)\varphi = -\frac{\lambda}{3!}\varphi^3.$$

Now the significance of calling the constant λ, a coupling is obvious. In the absence of this term, one has a linear equation, namely the Klein-Gordon equation,

$$\left(\Box + m^2\right)\varphi = 0.$$

This can be thought of as a free field because a linear system of partial differential equations has solutions which obey the superposition principle, namely, if φ_1 and φ_2 are two independent solutions, then the linear combination, $\varphi = c_1\varphi_1 + c_2\varphi_2$ is a solution as well. Physically this means the two waves/disturbances, φ_1 and φ_2 do not see other and "pass thru" each other unaffected. That is why despite the fact that the term $m^2\varphi^2$ appears in the potential energy, it is not thought of as a coupling or interaction since it does not lead to any real physical interactions (say between the solutions, φ_1 and φ_2). On the contrary, introduction of the $\lambda\varphi^4$ term renders the equation of motion for the field nonlinear and which means, one cannot have a linear superposition, $\varphi = c_1\varphi_1 + c_2\varphi_2$ to be a solution if φ_1 and φ_2 are two independent solutions. Physically speaking, the two waves/disturbances, φ_1 and φ_2, cannot pass through each other unaffected in the presence of the λ-term. Hence we are justified in calling λ a coupling constant and the associated term, $\lambda\varphi^4$, the coupling/interaction term.

1.1 Some dimensional analysis

Let's determine the (mass) dimensions of the field, φ, the constants in the potential, m, λ. To determine the dimensions of φ, we look at the kinetic term in the action,

$$I = \int d^4x\,\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \ldots$$

Since dimensions of both sides must match we look at dimensions of both sides. Before doing that recall that we are using natural units in which $\hbar = c = 1$, both are dimensionless $[\hbar] = [c] = 0$. Also recall that in natural units, length has negative dimensions, $[L] = -1$ while mass has dimensions, $[M] = 1$. Similarly one can show in natural units, [Linear Momentum] = [Energy] = +1, and [Angular momentum] = $[x \times p] = 0$. Thus in natural units, the action, which has same dimensions of angular momentum is dimensionless as well, $[I] = 0$. Thus the dimensions of the LHS of the above dimensional equation is zero. This means the RHS should have dimension zero as well. Let's look at the dimensions of the RHS. The integration measure, d^4x of course has dimensions as four powers of length, $d^4x \sim L^4$. Similarly the derivatives have dimensions of inverse length, $\partial \sim L^{-1}$. Thus we have,

$$\left[d^4x \, \frac{1}{2}\partial_\mu\varphi \, \partial^\mu\varphi\right] = \left[L^2\varphi^2\right] = 2\,[L] + 2\,[\varphi] = 2\,[\varphi] - 2.$$

Since the whole thing is dimensionless, one has,

$$[\varphi] = 1,$$

i.e. it has same dimensions as mass. These naive dimensions we are determining based on dimensional analysis are referred to the classical dimensions or *engineering dimensions* of φ.

Having determined the dimension of the field φ, it will now be easy to determine the engineering dimensions of the constants, m and λ. Again we start from the fact that the action is dimensionless, so the dimension of each term from the potential energy integral should vanish as well,

$$\left[\int d^4x \, \frac{m}{2} \, \varphi^2\right] = 0, \qquad \left[\int d^4x \, \frac{\lambda}{4!} \, \varphi^4\right] = 0.$$

These two conditions immediately imply,

$$[m] = +1, \qquad [\lambda] = 0.$$

Since the constant, m has dimensions of mass, we are justifying in referring the term $\frac{m}{2!}\varphi^2$ in the lagrangian as the *mass term*. Later on, when we quantize this theory we will see that indeed, m is (related to) the mass of the quantum of this field (particle). The quartic coupling constant, λ is dimensionless (in $3+1$ spacetime dimensions only, in higher or lower dimensions it will have non-zero dimensions).