Fall 2019: Classical Field Theory (PH6297)

Scalar Field Theory II: The free theory*

August 22, 2019

1 Free scalar field theory: Klein-Gordon equation

The free scalar field is described by the Klein-Gordon equation, namely,

$$\left(\Box + m^2\right) \varphi(x) = 0. \tag{1}$$

This equation is obtained as the Euler-Lagrange equation from varying the action,

$$I\left[\varphi\right] = \int d^4x \, \left(\frac{1}{2}\partial_\mu \varphi \, \partial^\mu \varphi - \frac{m}{2}\, \varphi^2\right).$$

Here we solve the Klein-Gordon equation. First note that if we set the parameter, $m = 0$, then the Klein-Gordon equation reduces to the wave equation, $\Box\varphi = 0$, which naturally will admit (plane) wave solutions. Anticipating a similar result for the Klein-Gordon equation we will first expand the field in a plane-wave basis, aka a four dimensional Fourier transform,

$$\varphi(x) = \int \frac{d^4x}{(2\pi)^4}\, \varphi(k)\, e^{-ik.x}. \tag{2}$$

Here, $k.x = k_\mu x^\mu$, and the four-dimensional wave-vector, k is defined by the components, $k^\mu = \left(\frac{\omega}{c}, \mathbf{k}\right)$. Since $x^\mu = (ct, \mathbf{x})$, we have $k.x = \omega t - \mathbf{k}\cdot\mathbf{x}$. From now on we will use natural units and omit all factors of speed of light, c. Since we are restricting ourselves to ***real*** scalar fields,

$$\varphi^*(x) = \varphi(x),$$

which implies one needs to impose the condition,

$$\varphi(-k) = \varphi^*(k).$$

Substituting this plane-wave basis expansion of $\varphi(x)$ in the Klein-Gordon equation, one gets,

$$\left(\Box + m^2\right) \int \frac{d^4x}{(2\pi)^4}\, \varphi(k)\, e^{-ik.x} = 0.$$

Since the only dependence on x is contained in the phase factor, one has,

$$\left(\Box + m^2\right) \int \frac{d^4x}{(2\pi)^4}\, \varphi(k)\, e^{-ik.x} = \int \frac{d^4x}{(2\pi)^4}\, \varphi(k)\, \left(\Box + m^2\right) e^{-ik.x}.$$

Next note that, $\partial_\mu(e^{-ik.x}) = -ik_\mu e^{-ik.x}$, i.e. a derivative acting on the phase-factor pulls down a factor of $-ik$ from the exponent, and one thus one has,

$$\int \frac{d^4x}{(2\pi)^4}\, \varphi(k)\, \left(\Box + m^2\right) e^{-ik.x} = \int \frac{d^4x}{(2\pi)^4}\, \varphi(k)\, \left(-k^2 + m^2\right) e^{-ik.x}.$$

*Notes for Lecture 10 (Aug. 21, 2019)

So the Klein-Gordon equation becomes,

$$\int \frac{d^4x}{(2\pi)^4}\, \varphi(k)\ \left(k^2 - m^2\right) e^{-ik.x} = 0.$$

Since the $e^{-ik.x}$ for different $k \in \mathbb{R}^4$ constitute a basis, for this equation to be valid one must have all basis coefficients vanishing, i.e.,

$$\varphi(k)\ \left(k^2 - m^2\right) = 0.$$

Next recall $k^2 = \omega^2 - \mathbf{k}^2$, where $\mathbf{k}^2 = \mathbf{k}\cdot\mathbf{k}$and using it one can rewrite the above equation as,

$$\varphi(k)\ \left(\omega^2 - \omega_{\mathbf{k}}^2\right) = 0, \tag{3}$$

where we have defined $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. This equation can be thought of as the momentum space (or Fourier space) version of Klein-Gordon equation. The solution to this equation, which is now an algebraic equation is very simple,

$$\varphi(k) \begin{cases} = 0, \omega^2 \neq \omega_{\mathbf{k}}^2 \\ \neq 0, \omega^2 = \omega_{\mathbf{k}}^2. \end{cases}$$

A better way of rewriting this is by means of the Dirac delta function,

$$\varphi(k) = \tilde{\varphi}(k)\ \delta(\omega^2 - \omega_{\mathbf{k}}^2).$$

The delta function can be further simplified using the identity,

$$\delta\left(f(x)\right) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}$$

where the x_i's are the roots of the equation, $f(x) = 0$. This implies,

$$\delta(\omega^2 - \omega_{\mathbf{k}}^2) = \frac{1}{2\omega_{\mathbf{k}}}\delta(\omega - \omega_{\mathbf{k}}) + \frac{1}{2\omega_{\mathbf{k}}}\delta(\omega + \omega_{\mathbf{k}}),$$

and,

$$\varphi(k) = \tilde{\varphi}(k)\ \delta(\omega^2 - \omega_{\mathbf{k}}^2) = \frac{1}{2\omega_{\mathbf{k}}}\tilde{\varphi}(k)\ \left(\delta(\omega - \omega_{\mathbf{k}}) + \delta(\omega + \omega_{\mathbf{k}})\right).$$

Plugging this back in the mode-expansion (2), one gets the general solution,

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{2\omega_{\mathbf{k}}}\tilde{\varphi}(k)\ \left(\delta(\omega - \omega_{\mathbf{k}}) + \delta(\omega + \omega_{\mathbf{k}})\right) e^{-ik.x}.$$

Next recall that, $d^4k = d\omega\, d^3\mathbf{k}$, and we can easily do the ω integral due to the delta function(s). Performing the ω-integral for both terms gives us the form of the general solution to be,

$$\varphi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^2}\,\frac{1}{(2\pi)\,(2\omega_{\mathbf{k}})}\tilde{\varphi}(\omega_{\mathbf{k}}, \mathbf{k})\ e^{-i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} + \int \frac{d^3\mathbf{k}}{(2\pi)^2}\,\frac{1}{(2\pi)\,(2\omega_{\mathbf{k}})}\tilde{\varphi}(-\omega_{\mathbf{k}}, \mathbf{k})\ e^{i(\omega_{\mathbf{k}}t + \mathbf{k}\cdot\mathbf{x})}.$$

Now we make a change of variable in the second term (integral), namely, $\mathbf{k} \to -\mathbf{k}$. Since $\omega_{\mathbf{k}}$ is an even function of $\mathbf{k}$, it is unaffected. However, $\tilde{\varphi}(-\omega_{\mathbf{k}}, \mathbf{k}) \to \tilde{\varphi}(-\omega_{\mathbf{k}}, -\mathbf{k})$ and $\mathbf{k}\cdot\mathbf{x} \to -\mathbf{k}\cdot\mathbf{x}$. Making these changes the general solution now looks like,

$$\varphi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^2}\,\frac{1}{(2\pi)\,(2\omega_{\mathbf{k}})}\left[\tilde{\varphi}(\omega_{\mathbf{k}}, \mathbf{k})\ e^{-i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} + \tilde{\varphi}(-\omega_{\mathbf{k}}, -\mathbf{k})\ e^{i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})}\right].$$

This is certainly real valued once we recall that,

$$\tilde{\varphi}(-\omega_{\mathbf{k}}, -\mathbf{k}) = \tilde{\varphi}^*(\omega_{\mathbf{k}}, \mathbf{k}).$$

Thus, we can write the final form of the most general solution of the Klein-Gordon equation in a plane wave basis to be,

$$\varphi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^2} \frac{1}{(2\pi)\,(2\omega_{\mathbf{k}})} \tilde{\varphi}(\omega_{\mathbf{k}}, \mathbf{k})\; e^{-i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})} + \mathrm{cc} \tag{4}$$

where "cc" stands for complex conjugate. The coefficients, $\tilde{\varphi}(\omega_{\mathbf{k}}, \mathbf{k})$ are completely arbitrary.

Thus the solutions to KG are indeed plane waves albeit with a dispersion relation,

$$\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}.$$

The phase velocity is,

$$c_{\mathbf{k}} = \frac{\omega_{\mathbf{k}}}{|\mathbf{k}|} = \sqrt{1 + \frac{m^2}{\mathbf{k}^2}}.$$

Thus these are dispersive waves, even in vacuo - different frequencies travel with different phase velocities. One might be alarmed by the fact that, $c_{\mathbf{k}} > 1$, i.e. phases travel faster than light and hence violate causality. However one must recall that while phases can travel faster than the speed of light, the whole wave-packet moves with the *group velocity*. We can compute the group velocity of the KG wave-packet,

$$v_g = \frac{d\omega_{\mathbf{k}}}{d\,|\mathbf{k}|} = \frac{|\mathbf{k}|}{\sqrt{\mathbf{k}^2 + m^2}}.$$

Thus $v_g < 1$ and information (energy-momentum) *does not* travel faster than light and there is no violation of causality!

2 The Yukawa meson field

Just as in the case of Maxwell's electrodynamics, one has propagating wave solutions in the absence of sources (charges and currents), we found in the absence of any sources (homogeneous equations) the real scalar field theory also admits wave solutions. Another important question in Maxwell theory is what is the electric field produced a source e.g. a point charge of strength q located at the point, $\mathbf{y}$. The answer takes the form of the famous Coulomb law,

$$\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi r^2}\hat{\mathbf{r}},$$

where $\mathbf{r} = \mathbf{x} - \mathbf{y}$. One might ask the same question for the real scalar theory, namely what is the expression for the field produced by a point source of strength g, located at position, $\mathbf{y}$, namely, a source density function

$$\rho(\mathbf{x}) = g\,\delta^3(\mathbf{x} - \mathbf{y}). \tag{5}$$

How do we include this source term in the real scalar theory? We will do this following Maxwell theory by inserting a potential term $\rho(x)\,\varphi(x)$ in the action (or lagrangian),

$$I\,[\varphi] = \int d^4x\, \left(\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m}{2}\,\varphi^2 - \rho\,\varphi\right),$$

with the specified source density (5). The equation of motion in this case is,

$$\left(\Box + m^2\right)\varphi(x) = -\rho(x).$$

Since the source (5) is time-independent (static), one can expect the field it creates to be also time independent (static), i.e. $\varphi(x) = \varphi(\mathbf{x})$, just a function of the spatial coordinates. In such a time-independent field, time-derivatives vanish : $\partial_t\varphi = 0$, and the equation reduces to a purely spatial equation,

$$\left(\nabla^2 - m^2\right)\varphi(\mathbf{x}) = g\,\delta^3(\mathbf{x} - \mathbf{y}). \tag{6}$$

This equation is reminiscent of the Coulomb Green's function equation in electrodynamics and we shall solve this equation using the same method we did for the Coulomb Green's function i.e. by Fourier transforming to momentum space,

$$\varphi(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, G(\mathbf{k})\, e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}. \tag{7}$$

Here $G(\mathbf{k})$ are the Fourier components of the field φ (we use the letter G because it is a Green's function i.e. for a delta function source). Plugging this in the LHS of equation (6) and on the RHS plugging the integral/Fourier representation of the Dirac delta function

$$\delta^3(\mathbf{x}-\mathbf{y}) = \varphi(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})},$$

one gets,

$$\left(\nabla^2 - m^2\right) \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, G(\mathbf{k})\, e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}) = g \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})},$$

or,

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3}\, G(\mathbf{k})\, \left(\nabla^2 - m^2\right) e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} = g \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}.$$

Next recall, $\boldsymbol{\nabla} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} = -i\mathbf{k}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$ and one has $\boldsymbol{\nabla}^2 \to (-i\mathbf{k})\cdot(-i\mathbf{k}) = -\mathbf{k}^2$ and the above equation becomes,

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3}\, G(\mathbf{k})\, \left(-\mathbf{k}^2 - m^2\right) e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, g\, e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}.$$

As before the $e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$'s form a basis and the only way the above equation can hold iff for all $\mathbf{k}$, the basis coefficients are equal on both sides, i.e.

$$G(\mathbf{k})\, \left(-\mathbf{k}^2 - m^2\right) = g,$$

which immediately give the Fourier components of the field to be,

$$G(\mathbf{k}) = -\frac{g}{\mathbf{k}^2 + m^2}. \tag{8}$$

Plugging this back in the mode expansion (7), we get the expression for the field

$$\varphi(\mathbf{x}) = -g \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \frac{1}{\mathbf{k}^2 + m^2}\, e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}.$$

Next we have to perform the momentum integrals. It will be convenient if we switch from Cartesian coordinates to spherical polar coordinates in momentum space, i.e. $\left(k^1, k^2, k^3\right) \to (q, \theta, \phi)$ where $q = \sqrt{|\mathbf{k}|}$. Further it will be even more convenient to choose, without any loss of generality the z-axis in the momentum space to be along the vector, $\mathbf{r} = \mathbf{x} - \mathbf{y}$. In these new coordinates the expression for the field is,

$$\varphi(\mathbf{x}) = -g \int \frac{q^2 dq\, \sin\theta d\theta\, d\phi}{(2\pi)^3}\, \frac{1}{q^2 + m^2}\, e^{-iqr\cos\theta}.$$

Here $r = |\mathbf{r}|$. The ϕ integral can be performed easily as nothing in the integrand depends on it. So we replace $\int d\phi = 2\pi$, and get,

$$\varphi(\mathbf{x}) = -\frac{g}{4\pi^2} \int_0^\infty dq \frac{q^2}{q^2 + m^2} \int_0^\pi d\theta\, \sin\theta\, e^{-iqr\cos\theta}.$$

Next the θ-integral is performed, namely,

$$\int_0^\pi d\theta\, \sin\theta\, e^{-iqr\cos\theta} = \int_{-1}^1 d(\cos\theta)\, e^{-iqr(\cos\theta)} = \frac{e^{iqr} - e^{-iqr}}{iqr}.$$

After this one has,

$$
\begin{aligned}
\varphi(\mathbf{x}) &= -\frac{g}{4\pi^2}\int_0^\infty dq \frac{q^2}{q^2+m^2}\,\frac{e^{iqr}-e^{-iqr}}{iqr} \\
&= \frac{g}{4\pi^2 r}\int_0^\infty dq \frac{iq}{q^2+m^2}\,\left(e^{iqr}-e^{-iqr}\right) \\
&= \frac{g}{4\pi^2 r}\left(\int_0^\infty dq \frac{iq}{q^2+m^2}\,e^{iqr} - \int_0^\infty dq \frac{iq}{q^2+m^2}\,e^{-iqr}\right).
\end{aligned}
$$

Now on the second integral we perform a change of variables, $q \to -q$. Under this change of variables the second integral becomes,

$$
-\int_0^\infty dq \frac{iq}{q^2+m^2}\,e^{-iqr} = \int_{-\infty}^0 dq \frac{iq}{q^2+m^2} e^{iqr}.
$$

which is the same integral as the first term except the range is now $(-\infty, 0)$. Summing these two contributions, the field expression becomes a single integral ranging from $(-\infty, \infty)$,

$$
\varphi(\mathbf{x}) = \frac{g}{4\pi^2 r}\int_{-\infty}^\infty dq \frac{iq}{q^2+m^2}\,e^{iqr}.
$$

Since $\frac{d}{dr}(e^{iqr}) = iq\,e^{iqr}$, we can rewrite the above expression as,

$$
\varphi(\mathbf{x}) = \frac{g}{4\pi^2 r}\frac{d}{dr}\left(\int_{-\infty}^\infty dq \frac{e^{iqr}}{q^2+m^2}\right). \tag{9}
$$

To get to the final form of the field one has to perform the q-integral, namely,

$$
J(r) = \int_{-\infty}^\infty dq \frac{e^{iqr}}{q^2+m^2}.
$$

We note that the integrand has two poles on the imaginary q-axis, , while the integration contour is along the real q-axis from $-\infty$ to ∞. To evaluate the integral, J using the residue theorem one has to close the contour, and in the process encircle either of the two poles, i.e. one can choose to close the contour from above (in the upper half-complex q-plane) anticlockwise, or close the contour from below (in the lower half complex q-plane) clockwise as shown in figure (1). The choice of the contour is determined by the physical boundary conditions, one must have

$$
\varphi(\mathbf{x}), J(r) \to 0, \text{ as } r \to \infty. \tag{10}
$$

This physically means the field weakens and becomes zero as one moves away from the location of the source to infinitely far distance. This boundary condition selects the anticlockwise contour in the upper half complex q-plane which encircles the pole, $q = +im$. Then using the residue theorem one has

$$
J = \oint dq \frac{e^{iqr}}{q^2+m^2} = \frac{2\pi i\, e^{i(im)r}}{2(im)} = \frac{\pi}{m}e^{-mr}.
$$

Thus indeed we find $J(r) \to 0$ as $r \to \infty$ [1]. Plugging this result back in (9), we get the final form of the scalar field produced by a point source of strength g located at $\mathbf{y}$,

$$
\varphi(\mathbf{x}) = -g\frac{e^{-mr}}{4\pi r},\ r = |\mathbf{x}-\mathbf{y}|\,. \tag{11}
$$

Although the factor $\frac{1}{4\pi r}$ is perhaps familiar from the Coulomb Green's function for electrodynamics for a unit point charge (source), there are a couple massive differences. First is the factor e^{-mr}, an exponential

[1] Had we selected the other contour, closing it from down in the lower half of complex q-plane, one would get $J(r) \propto e^{mr}$ thus leading to $J(r) \to \infty$ as $r \to \infty$. This would have been unphysical as the field (effect of the source) grows larger as we move farther from the source.

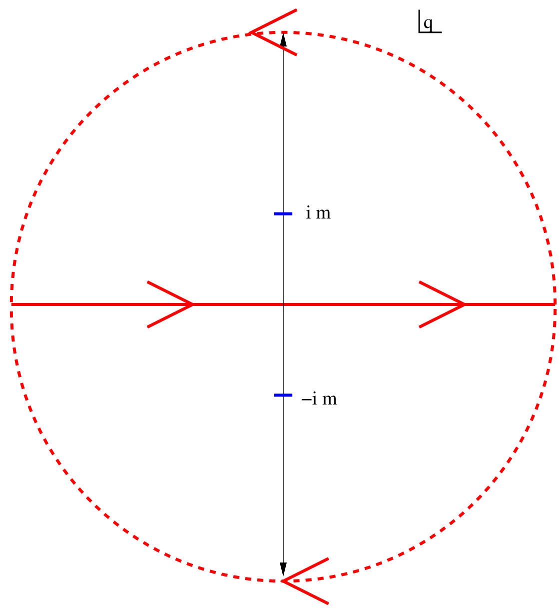

Figure 1: Two possible contours for the Yukawa meson field (Green's function) in the complex q-plane

suppression of the field as one goes away from the source. This effectively means the effect of the sources dissipates to negligible amount $r > \frac{1}{m}$. Thus unlike electromagnetic force, the Klein-Gordon scalar field represents a **short-range interaction (force)**. Second, the overall negative sign implies that the force is an attractive one. The best way to see this is to ask what is the potential energy when we introduce a test source $\rho(\mathbf{x}) = \varepsilon\, \delta^3(\mathbf{x} - \mathbf{x}')$ of very weal strength ε in the field (11). The interaction energy is,

$$U = \int d^3\mathbf{x}\, \rho(\mathbf{x})\varphi(\mathbf{x}) = -g\varepsilon\, \frac{e^{-m|\mathbf{x}'-\mathbf{y}|}}{|\mathbf{x}' - \mathbf{y}|} < 0.$$

Since the potential energy is negative, this implies an **attractive interaction**. Such a real scalar field was proposed to be the force-field responsible for holding two protons or two neutrons together in the nucleus of an atom by H. Yukawa in 1935. That real scalar field went by the name of Yukawa's meson (field). From the range of the nuclear force, i.e. $\sim 10^{-15}m$, he was able to predict the mass parameter, $m \sim 10^{2-3}$ times the mass of the electron. On quantizing the Yukawa meson field thus one should expect the quanta of the meson field, namely the meson particles to have appear in nature with the same mass. Indeed in 1947, such meson particles were observed by Powell in cosmic rays.

However this real scalar is electrically neutral and cannot describe the interaction (force) between the neutron and the proton. For that one needs a theory of (electrically) charged scalars and the correct theory for that would necessarily involve a *complex* scalar field. We will take up the complex scalar field theory in the next lecture.

Homework: Show that the choice of the boundary condition (10) in real space is equivalent to the choice of the counterclockwise contour in the upper half complex q-plane i.e. momentum space.